

Superior discretizations and AMG solvers for extremely anisotropic diffusion via hyperbolic operators

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Outline

Anisotropic diffusion

Spatial discretization

- Discontinuous elements

- Continuous elements

- Structure preservation

Linear solvers

- AIR Algebraic Multigrid

- Open field lines (DG)

- Closed field lines

Anisotropic diffusion

Evolution of temperature T in plasma, heat flux $\mathbf{q}$, forcing S :

$$\frac{\partial T}{\partial t} - \nabla \cdot (\kappa_{\parallel} \nabla_{\parallel} + \kappa_{\perp} \nabla_{\perp}) T = S,$$

(Dirichlet BCs) $T|_{\partial\Omega} = T_{\text{BC}},$

where $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ is direction of magnetic field lines, and

$$\nabla_{\parallel}(\cdot) := \mathbf{b} \cdot \nabla(\cdot) \mathbf{b},$$
$$\nabla_{\perp} := \nabla - \nabla_{\parallel}.$$

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Many ways to skin a PDE!

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- In magnetic confinement fusion, expect $\frac{\kappa_{\parallel}}{\kappa_{\perp}} \sim 10^9 - 10^{10}$.
- Parabolic equation in nature, but for non-grid-aligned $\mathbf{b}$, both discretization accuracy and implicit solve are very hard.

Directional gradient formulation

Define purely anisotropic conductivity $\kappa_{\Delta} := \kappa_{\parallel} - \kappa_{\perp}$.

- Diffusion isotropic if $\kappa_{\Delta} = 0$,
- Predominately anisotropic if $\kappa_{\Delta} \gg \kappa_{\perp}$.

Heat flux can be written as

$$\mathbf{q} = \kappa_{\Delta} \nabla_{\parallel} T + \kappa_{\perp} \nabla T.$$

Use directional derivative $\mathbf{b} \cdot \nabla T$ as auxiliary variable, reformulate

$$\begin{aligned} \frac{\partial T}{\partial t} - \sqrt{\kappa_{\Delta}} \nabla \cdot (\mathbf{b} \zeta) - \nabla \cdot (\kappa_{\perp} \nabla T) &= S, \\ \zeta &= \sqrt{\kappa_{\Delta}} \mathbf{b} \cdot \nabla T. \end{aligned}$$

The steady state limit

Written in block operator form, directional gradient formulation is

$$\begin{pmatrix} \frac{\partial}{\partial t} + \kappa_{\perp} \Delta & -\sqrt{\kappa_{\Delta}} \nabla \cdot (\mathbf{b}(\cdot)) \\ -\sqrt{\kappa_{\Delta}} \mathbf{b} \cdot \nabla & I \end{pmatrix} \begin{pmatrix} T \\ \zeta \end{pmatrix} = \begin{pmatrix} S \\ 0 \end{pmatrix}.$$

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Consider steady-state (no time derivative) and purely anisotropic problem ($\kappa_{\perp} = 0$). Rearranging system, we have

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$\Rightarrow$ In the regime of extreme anisotropy, I claim we should treat this equation as a system of coupled hyperbolic operators rather than a parabolic operator.

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Upwind DG discretization

Discretize the transport term $\nabla \cdot (\sqrt{\kappa_\Delta} \mathbf{b}(\cdot))$, using classical DG-upwind method:

$$L_b(\theta; \phi) := - \langle \sqrt{\kappa_\Delta} \theta \mathbf{b}, \nabla \phi \rangle + \int_{\Gamma} \llbracket \sqrt{\kappa_\Delta} \phi \mathbf{b} \cdot \mathbf{n} \rrbracket \tilde{\theta} \, dS + \int_{\partial\Omega_{\text{out}}} \sqrt{\kappa_\Delta} \phi \mathbf{b} \cdot \mathbf{n} \theta \, dS,$$

for L^2 -inner product $\langle \cdot, \cdot \rangle$, and any functions $\theta, \phi \in \mathbb{V}_k^{\text{DG}}(\Omega)$. $\partial\Omega_{\text{out}}$ denotes the outflow subset of boundary $\partial\Omega$ relative to $\mathbf{B}$, Γ denotes all interior facets, and

$$\llbracket \psi \rrbracket := \psi^+ - \psi^-, \quad \tilde{\psi} := \begin{cases} \psi^+ & \text{if } \mathbf{b}^+ \cdot \mathbf{n}^+ < 0, \\ \psi^- & \text{otherwise.} \end{cases}$$

Full weak form

Define two transport operators

$$L_{b,T}(\zeta_h; \phi) := L_b(\zeta_h; \phi) + \int_{\partial\Omega_{\text{in}}} \sqrt{\kappa_\Delta} \phi(\mathbf{b} \cdot \mathbf{n}) \zeta_{\text{in}} \, dS \quad \forall \phi \in \mathbb{V}_k^{\text{DG}},$$

$$L_{b,\zeta}(\psi; T_h) := L_b(\psi; T_h) - \int_{\partial\Omega_{\text{out}}} \sqrt{\kappa_\Delta} T_{\text{BC}}(\mathbf{b} \cdot \mathbf{n}) \psi \, dS \quad \forall \psi \in \mathbb{V}_k^{\text{DG}},$$

where $\partial\Omega_{\text{in}}$ denotes inflow boundary, ζ_{in} is known.

$\implies$ *Known inflow boundary ensure $L_{b,T}$, $L_{b,\zeta}$ are invertible*

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Two options for ζ_{in} :

1. Iterate implicit solve until $\zeta_{\text{in}} = \zeta_h$ (within nonlinear iteration for T , this is almost certainly less stiff and easier to resolve)
2. Treat auxiliary inflow ζ_{in} explicitly; so far have found in tests it varies slowly in time.

Full weak form

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where $\partial\Omega_{\text{in}}$ denotes inflow boundary, ζ_{in} is known.

Weak form: find $(T_h, \zeta_h) \in (\mathbb{V}_k^{\text{DG}} \times \mathbb{V}_k^{\text{DG}})$ such that

$$\left\langle \phi, \frac{\partial T_h}{\partial t} \right\rangle - L_{b,T}(\zeta_h; \phi) - \text{IP}(T_h; \phi) = - \int_{\partial\Omega} \kappa_{\text{BC}} \phi (T_h - T_{\text{BC}}) \, dS + \langle \phi, \mathbf{S} \rangle,$$
$$\langle \psi, \zeta_h \rangle + L_{b,\zeta}(\psi; T_h) = 0 \quad \forall \psi, \phi \in \mathbb{V}_k^{\text{DG}}.$$

MHD test problem

Domain $\Omega = [0, 1] \times [0, 1] \times [0, 5]$, periodic in z , initial T

$$T_0 = \sin\left(\frac{\pi x}{L_x}\right) \sin\left(\frac{\pi y}{L_x}\right),$$

with $T_{BC} = 0$ on $\partial\Omega$. $\mathbf{B}$ aligns with contours of T_0 in (x, y) and is constant in z ,

$$\mathbf{B} = (B_x, B_y, B_z) = (-\partial_y T, \partial_x T, 5)$$

$\implies$ ensures $(B_x, B_y) = \nabla^\perp T_0$ for 2D curl $\nabla^\perp = (-\partial_y, \partial_x)$.

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Add counter forcing $S = -\kappa_\perp \Delta T_0$, to ensure a steady state test case. Error can be measured relative to the initial condition,

$$e_T(t) = \frac{\|T_h(t) - T_0\|_2}{\|T_0\|_2}.$$

Accuracy and results

Compare with:

- **CG-primal system:** (i.e. without auxiliary heat flux) for T in second order CG, $\mathbb{V}_2^{\text{CG}}$.
- **DG-primal system:** for T in $\mathbb{V}_2^{\text{DG}}$ (similar to recent papers Green et al, Vogl et al.).
- **Mixed CG:** original directional gradient formulation in $\mathbb{V}_2^{\text{CG}} - \mathbb{V}_1^{\text{DG}}$ from Gunter et al (does not use transport disc. techniques).

Accuracy and results

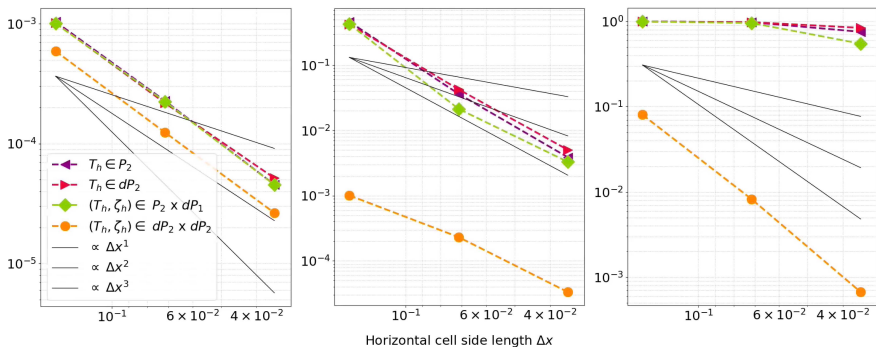


Figure: Relative L^2 error for anisotropy ratios from left to right 10^3 , 10^6 , and 10^9 . Orange circles denote novel mixed DG scheme, green diamonds mixed CG, red triangles primal DG, and purple triangles primal CG.

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Continuous Galerkin

Let $\mathbb{V}_T := CG(\Omega)_k$ denote k^{th} order continuous Galerkin space, and

$$\mathbb{V}_{T_{bc}} := \{\eta \in CG(\Omega)_k : \eta|_{\partial\Omega} = T_{bc}\}, \quad \mathring{\mathbb{V}}_T := \{\eta \in CG(\Omega)_k : \eta|_{\partial\Omega} = 0\}.$$

Weak form: find $(T \in \mathbb{V}_{T_{bc}}, \zeta \in \mathbb{V}_\zeta)$ such that

$$\left\langle \eta, \frac{\partial T}{\partial t} \right\rangle + \langle \kappa_{||} \mathbf{b} \cdot \nabla \eta, \zeta \rangle = F \quad \forall \eta \in \mathring{\mathbb{V}}_T, \quad (1)$$

$$\langle \eta, \zeta - \mathbf{b} \cdot \nabla T \rangle = 0 \quad \forall \phi \in \mathbb{V}_\zeta. \quad (2)$$

for CG or DG auxiliary space $\mathbb{V}_\zeta$ w/o any associated BCs.

- Analogous to DG case, we set auxiliary space equal to temperature space (ensures square advective blocks).

Transport stabilization (it works!)

Define velocity field $\mathbf{s} := \sqrt{\kappa_{\parallel}} \mathbf{b}$. For stabilization parameter τ , define advective and flux based SUPG operators

$$S_a(\eta) = \eta + \tau \mathbf{s} \cdot \nabla \eta, \quad S_f(\eta) = \eta + \mathbf{s} \cdot \nabla (\tau \eta).$$

Mixed CG, SUPG stabilized discretization given by

$$\begin{aligned} \left\langle S_a(\eta), \frac{\partial T}{\partial t} \right\rangle + \langle \mathbf{s} \cdot \nabla \eta, S_f(\zeta) \rangle &= \langle S_a(\eta), F \rangle + \\ &\quad \int_{\partial\Omega} \tau (\mathbf{s} \cdot \nabla \eta) (\mathbf{s} \cdot \mathbf{n}) (\mathbf{s} \cdot \nabla T) dS \quad \forall \eta \in \mathring{\mathbb{V}}_T, \\ \langle \eta, S_f(\zeta) \rangle - \langle \mathbf{s} \cdot \nabla T, S_f(\eta) \rangle &= 0 \quad \forall \eta \in \mathbb{V}_T. \end{aligned}$$

for temperature $T \in \mathbb{V}_{T_{bc}}$ and auxiliary variable $\zeta \in \mathbb{V}_T$.

Transport stabilization (it works!)

- A boundary integral appears because while η vanishes along Dirichlet boundary, SUPG-modification $\mathbf{s} \cdot \nabla \eta$ need not (gradient first computed within cells neighboring $\partial\Omega$, then evaluated along $\partial\Omega$)
- Averaged stabilization parameter for tuning parameter $\lambda \in [0, 1]$:

$$\tau = \left(\lambda \frac{2}{\Delta t} + \frac{\sqrt{\kappa_{\parallel}}}{2\Delta x} \right)^{-1}$$

- Specific form of $S_f(\eta)$ and $S_a(\eta)$ derived for structure preservation and consistency.
- In practice, we use weak BCs for CG formulation. Recall we swap the algebraic system's block rows. In Firedrake, non-trivial to form the correct block row-swapped system with strong BCs.

Accuracy (quads)

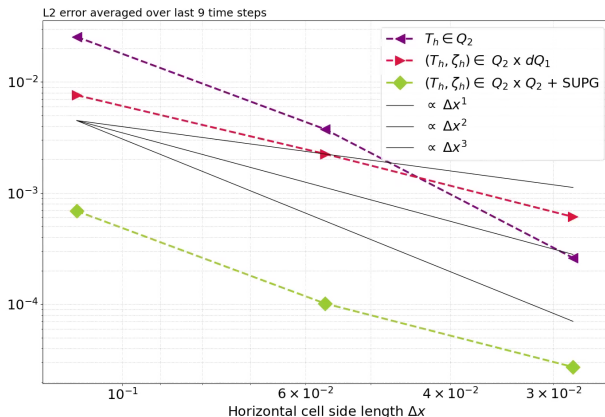


Figure: Magnetic surface test case: plasma trapped in series of concentric tori. We “unfold” concentric tori with rational winding number and temperature perturbation spreading on surface. $\kappa_{\perp} = 0, \kappa_{\parallel} = 100$, integrate to one Alfvén time.

Accuracy (triangles)

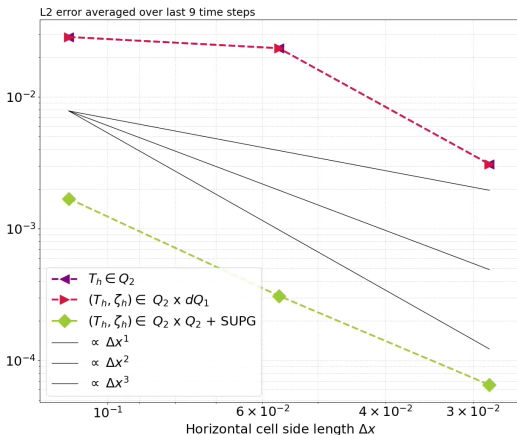


Figure: Magnetic surface test case; “unfolded” concentric tori with rational winding number and temperature perturbation spreading on surface.

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Consistency

Proposition: *The CG and DG space discretizations are consistent, i.e. a solution satisfying the continuous equation also satisfies the discretizations with sufficiently regular test functions η .*

Note, most discretizations would be consistent, but using stabilization techniques like hyper-diffusion are not.

Diffusion

Continuous system is dissipative up to up to forcing and total in- and outflux:

$$\frac{1}{2} \frac{d}{dt} \|T\|_2^2 = \langle T, F \rangle - \|\sqrt{\kappa_{\parallel}} \mathbf{b} \cdot \nabla T\|_2^2 + \int_{\partial\Omega} T \kappa_{\parallel} (\mathbf{b} \cdot \mathbf{n}) (\mathbf{b} \cdot \nabla T) dS.$$

- DG discretization satisfies this weakly, in the sense that the diffusion property is satisfied up to a weak BC penalty, and if a strong solution satisfies the BCs, we satisfy the diffusion property.
- CG discretization with Neumann or weak Dirichlet BCs satisfies this weakly.

Energy/Temperature

Continuous system conserves total temperature up to forcing and total in- and outflux:

$$\frac{d}{dt} \int_{\Omega} T \, dx = \int_{\Omega} F \, dx + \int_{\partial\Omega} \kappa_{\parallel} (\mathbf{b} \cdot \mathbf{n}) (\mathbf{b} \cdot \nabla T) dS.$$

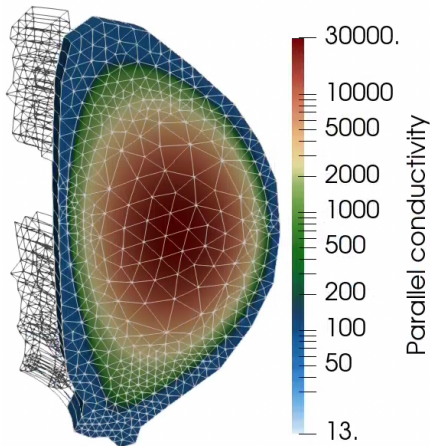
- DG discretization satisfies this weakly up to normal continuity $\mathbf{b}$ (strong solution will satisfy).
- CG discretization with Neumann BCs satisfies this automatically.
- CG discretization with weak Dirichlet BCs satisfies this weakly (multiple extra terms in weak form, but eliminated with strong solution).

Spurious heat loss

Movie!

Spurious heat loss

- $\kappa_{\perp} = 0$, physical $\kappa_{\parallel}$ taken from Braginski model, fixed Dirichlet BCs.
- Time step of 1 Alfven time.
- Periodic domain in angle, CG disc. has 11K DOFs, DG disc. 47K DOFs.



Spurious heat loss

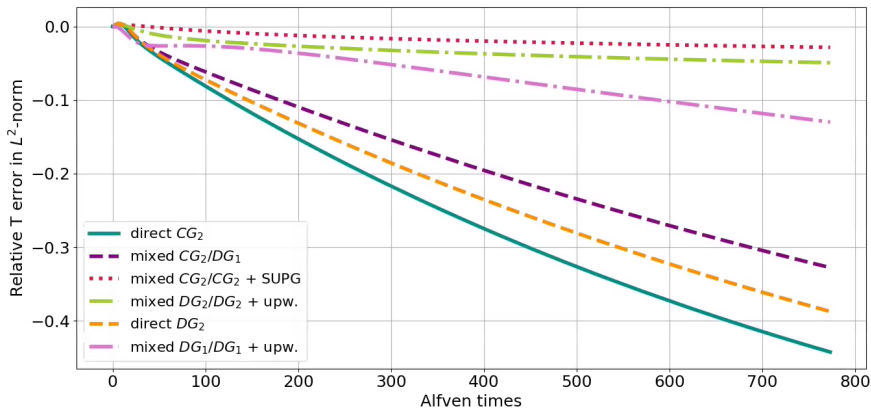


Figure: Spurious temperature loss for different discretizations. Final losses: **CG2: -44%**; **DG2: -39%**; **CG2-DG1: -33%**; **DG1-DG1: -13%**; **DG2-DG2: -4.9%**; **CG2-CG2: -2.8%**.

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Algebraic multigrid (AMG)

One of fastest methods (when applicable) to solve large, sparse systems $A\mathbf{x} = \mathbf{b}$.

$\implies$ Two parts: *Relaxation* and *coarse-grid correction*:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + M^{-1}(\mathbf{b} - A\mathbf{x}_k),$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + P(RAP)^{-1}R(\mathbf{b} - A\mathbf{x}_k).$$

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$$\begin{aligned}\mathbf{e}_{k+1} &= (I - M^{-1}A)\mathbf{e}_k, \\ \mathbf{e}_{k+1} &= (I - P(RAP)^{-1}RA)\mathbf{e}_k \\ &= (I - \Pi)\mathbf{e}_k.\end{aligned}$$

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AMG is typically for symmetric positive definite (SPD) matrices.

- If A is SPD, $\|\mathbf{x}\|_A^2 = \langle A\mathbf{x}, \mathbf{x} \rangle$ defines norm.
- $R := P^T \implies \|\Pi\|_A = 1$.

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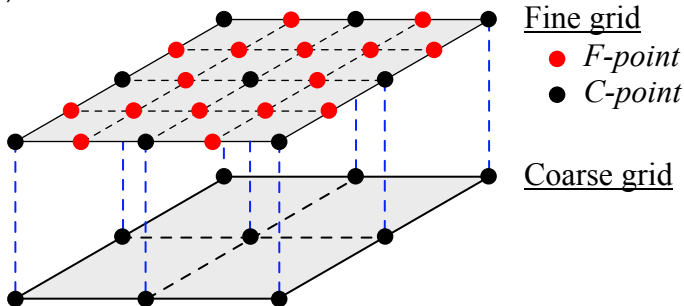
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- **Non-orthogonal projection can increase error ☹**

Notation

CF-Splitting:

- Assume DOFs partitioned into coarse points (C) and fine points (F)



Notation

CF-Splitting:

- Assume DOFs partitioned into coarse points (C) and fine points (F)

Write vectors and matrices in block form:

$$\begin{aligned} A &= \begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix}, & P &= \begin{pmatrix} W \\ I \end{pmatrix}, \\ R &= \begin{pmatrix} Z & I \end{pmatrix}, & \mathcal{K} &:= RAP. \end{aligned}$$

Coarse-grid correction and R_{ideal}

Decompose F-point error $\mathbf{e}_f = W\mathbf{e}_c + \delta\mathbf{e}_f$:

$$\mathbf{e}^{(i+1)} = \begin{pmatrix} \mathbf{e}_f^{(i)} \\ \mathbf{e}_c^{(i)} \end{pmatrix} - P(RAP)^{-1}RA \left[\begin{pmatrix} W\mathbf{e}_c^{(i)} \\ \mathbf{e}_c^{(i)} \end{pmatrix} + \begin{pmatrix} \delta\mathbf{e}_f^{(i)} \\ \mathbf{0} \end{pmatrix} \right]$$

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Ideal restriction:

$$\begin{aligned}RA \begin{pmatrix} \delta\mathbf{e}_f^{(i)} \\ \mathbf{0} \end{pmatrix} &= (ZA_{ff} + A_{cf})\delta\mathbf{e}_f = \mathbf{0} \\ \implies \underline{R_{\text{ideal}} = (-A_{cf}A_{ff}^{-1} \quad I)}\end{aligned}$$

Coarse-grid correction and R_{ideal}

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Lemma 1 (Orthogonal coarse-grid correction).

Coarse-grid correction with

$$R_{\text{ideal}} = \begin{pmatrix} -A_{cf}A_{ff}^{-1} & I \end{pmatrix}, \quad P = \begin{pmatrix} \mathbf{0} \\ I \end{pmatrix}$$

is an ℓ^2 -orthogonal projection.

Reduction based AMG as block LDU

Partition (discontinuous) elements into C-elements and F-elements.
Then in matrix form,

$$\begin{bmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{bmatrix}^{-1} = \begin{bmatrix} I & -A_{ff}^{-1}A_{fc} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{ff}^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{cf}A_{ff}^{-1} & I \end{bmatrix}.$$

Reduction based AMG preconditioner M^{-1} looks like:

$$M^{-1} = \begin{bmatrix} I & \widehat{W} \\ 0 & I \end{bmatrix} \begin{bmatrix} \Delta_F & 0 \\ 0 & \mathcal{K}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix},$$

where $\widehat{W} = (I - \Delta_F A_{ff})W - \Delta A_{fc}$. Want $\Delta_F \simeq A_{ff}^{-1}$, $Z \simeq -A_{cf}A_{ff}^{-1}$.

Reduction based AMG as block LDU

Partition (discontinuous) elements into C-elements and F-elements.
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where $\widehat{W} = (I - \Delta_F A_{ff})W - \Delta A_{fc}$. Want $\Delta_F \simeq A_{ff}^{-1}$, $Z \simeq -A_{cf}A_{ff}^{-1}$.

$\implies$ Can we approximate A_{ff}^{-1} well?

Approximate ideal restriction

For i th C-point (i th row of R), choose restriction neighborhood $\mathcal{R}_i = \{\ell_1, \dots, \ell_{S_i}\}$ of some “nearby” F-points. Solve

$$a_{ij} + \sum_{k \in \mathcal{R}_i} z_{ik} a_{kj} = 0.$$

Sets RA equal to zero within F-point sparsity pattern.

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$$\begin{pmatrix} a_{\ell_0 \ell_0} & a_{\ell_1 \ell_0} & \dots & a_{\ell_{S_i} \ell_0} \\ a_{\ell_0 \ell_1} & a_{\ell_1 \ell_1} & \dots & a_{\ell_{S_i} \ell_1} \\ \vdots & & \ddots & \vdots \\ a_{\ell_0 \ell_{S_i}} & a_{\ell_1 \ell_{S_i}} & \dots & a_{\ell_{S_i} \ell_{S_i}} \end{pmatrix} \begin{pmatrix} z_{i \ell_0} \\ z_{i \ell_1} \\ \vdots \\ z_{i \ell_{S_i}} \end{pmatrix} = - \begin{pmatrix} a_{i \ell_0} \\ a_{i \ell_1} \\ \vdots \\ a_{i \ell_{S_i}} \end{pmatrix}$$

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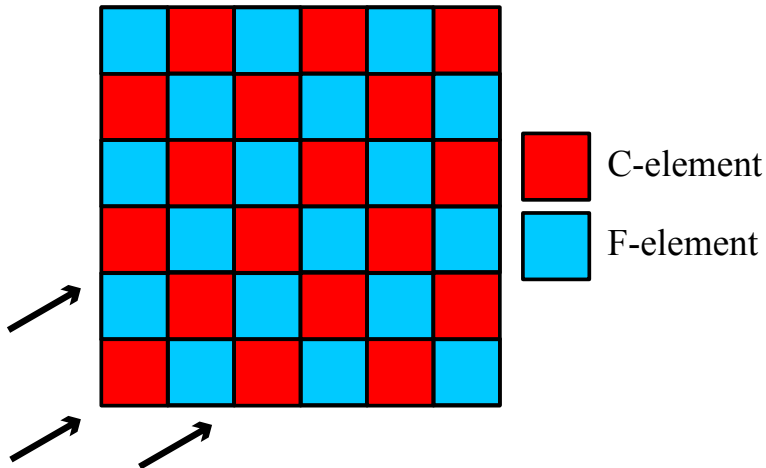
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Goal:

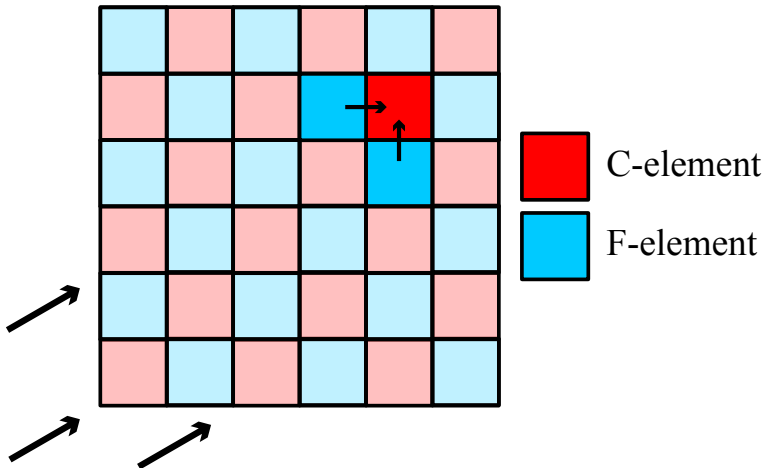
1. Use AIR to achieve accurate solution at C-points.
2. Follow with F-relaxation to distribute accuracy to F-points.

AIR for upwind transport



Consider transport on structured 2d grid and partition elements into C-elements and F-elements.

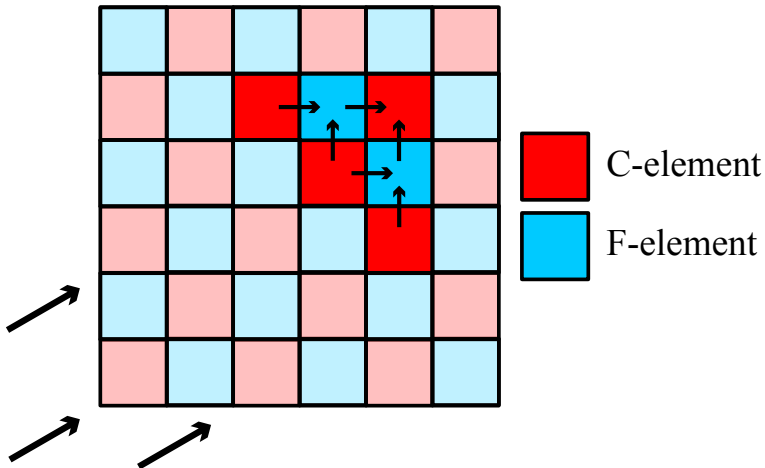
AIR for upwind transport



Notice that there are no C-C or F-F connections

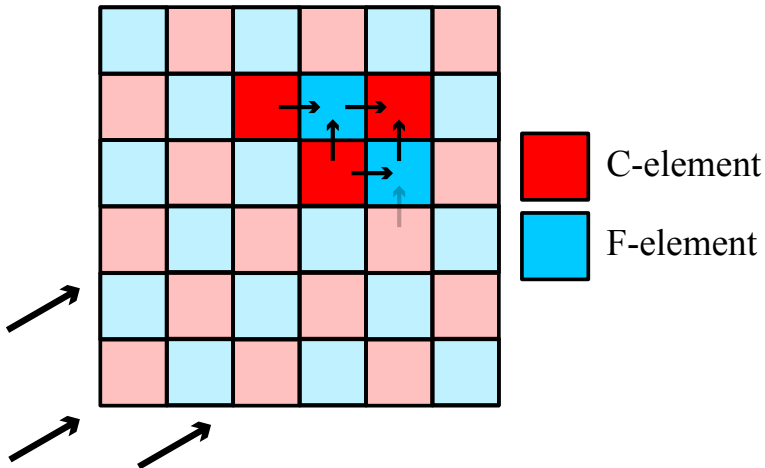
$$\implies A_{ff} = A_{cc} = I.$$

AIR for upwind transport



If $A_{ff} = I$, AMG coarse grid given by $A_{cc} - A_{cf}A_{fc} \iff$ all C-F-C connections.

AIR for upwind transport



If $A_{ff} = I$, AMG coarse grid given by $A_{cc} - A_{cf}A_{fc} \iff$ all C-F-C connections. **One of these connections is weak!**

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2×2 DG system

Reorder discrete 2×2 block system

$$\begin{pmatrix} -\sqrt{\kappa_{\Delta}} G_b & M \\ \frac{1}{\Delta t}(M + \tilde{\kappa}_{BC} M_{BC, h_e}) + \kappa_{\perp} L & \sqrt{\kappa_{\Delta}} G_b^T \end{pmatrix} \begin{pmatrix} T^{n+1} \\ \zeta^{n+1} \end{pmatrix} = \begin{pmatrix} F_{\zeta} \\ F_T \end{pmatrix}.$$

Assume $\tilde{\kappa}_{BC} \sim \mathcal{O}(1)$, so that $M + \tilde{\kappa}_{BC} M_{BC, h_e} \approx M$ in spectral analysis. Apply block triangular preconditioner,

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Convergence of fixed-point / Krylov fully defined by approximating Schur complement:

$$S_{22} := \sqrt{\kappa_{\Delta}} G_b^T + \left(\frac{1}{\Delta t} M + \kappa_{\perp} L \right) (\sqrt{\kappa_{\Delta}} G_b)^{-1} M.$$

Spectral analysis of Schur complement

Preconditioned S_{22} similar to SPD operator:

$$\begin{aligned}(\sqrt{\kappa_\Delta} G_b^T)^{-1} S_{22} &= I + \frac{1}{\Delta t \kappa_\Delta} G_b^{-T} M G_b^{-1} M + \frac{\kappa_\perp}{\kappa_\Delta} G_b^{-T} L G_b^{-1} M \\ &\sim I + \frac{1}{\Delta t \kappa_\Delta} M^{1/2} G_b^{-T} M G_b^{-1} M^{1/2} + \frac{\kappa_\perp}{\kappa_\Delta} M^{1/2} G_b^{-T} L G_b^{-1} M^{1/2}.\end{aligned}$$

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First (anisotropic) term: for mesh h and constants c_1, c_2 :

$$\sigma\left(\frac{1}{\Delta t \kappa_\Delta} M^{1/2} G_b^{-T} M G_b^{-1} M^{1/2}\right) \subset \frac{1}{\Delta t \kappa_\Delta} [c_1 h^2, c_2].$$

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Second (isotropic) term: 2D and $\mathbf{b}$ aligned in x . Second term $\sim \partial_{xx}^{-1}(\partial_{xx} + \partial_{yy}) = 1 + \partial_{xx}^{-1} \partial_{yy}$. On unit domain, Laplacian eigendecomposition $\{u_{jk}, \pi^2(j^2 + k^2)\}$, $u_{jk} = 2 \sin(j\pi x) \sin(k\pi y)$; similar for ∂_{xx} in (j, x) . $\implies$ Highest frequency in y , $k = 1/h$, and smoothest in x , $j = 1$, yields eigenpair

$$(1 + \partial_{xx}^{-1} \partial_{yy}) 2 \sin(\pi x) \sin(\frac{1}{h} \pi y) = (1 + \frac{1}{h^2}) 2 \sin(\pi x) \sin(\frac{1}{h} \pi y).$$

Spectral analysis of Schur complement

Preconditioned S_{22} similar to SPD operator:

$$\begin{aligned} (\sqrt{\kappa_\Delta} G_b^T)^{-1} S_{22} &= I + \frac{1}{\Delta t \kappa_\Delta} G_b^{-T} M G_b^{-1} M + \frac{\kappa_\perp}{\kappa_\Delta} G_b^{-T} L G_b^{-1} M \\ &\sim I + \frac{1}{\Delta t \kappa_\Delta} M^{1/2} G_b^{-T} M G_b^{-1} M^{1/2} + \frac{\kappa_\perp}{\kappa_\Delta} M^{1/2} G_b^{-T} L G_b^{-1} M^{1/2}. \end{aligned}$$

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Second (isotropic) term:

$$\sigma \left(\frac{\kappa_\perp}{\kappa_\Delta} M^{1/2} G_b^{-T} L G_b^{-1} M^{1/2} \right) \subset \left(0, \frac{\kappa_\perp c_3}{\kappa_\Delta h^2} \right],$$

Spectral analysis of Schur complement

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$$\sigma\left((\sqrt{\kappa_\Delta} G_b^T)^{-1} S_{22}\right) \subset \left(1, 1 + \frac{\kappa_\perp c_3}{\kappa_\Delta h^2} + \frac{c_2}{\Delta t \kappa_\Delta}\right].$$

Spectral analysis of Schur complement

Eigenvalues computed in practice via Lanczos:

Δt	$\kappa_{\parallel}/\kappa_{\perp}$	2D			3D		
10^{-3}	10^3	7.8e0	4.7e0	4.3e0	6.6e2	4.0e2	3.4e2
	10^6	7.8e-3	4.7e-3	4.3e-3	6.6e-1	4.0e-1	3.4e-1
	10^9	7.8e-6	4.7e-6	4.3e-6	6.6e-4	4.0e-4	3.4e-4
10^{-2}	10^3	8.0e-1	1.5e0	3.7e0	6.6e1	1.1e2	2.8e2
	10^6	8.0e-4	1.5e-3	3.7e-3	6.6e-2	1.1e-1	2.8e-1
	10^9	8.0e-7	1.5e-6	3.7e-6	6.6e-5	1.1e-4	2.8e-4

Table: Largest eigenvalues of error propagation, $(\sqrt{\kappa_{\Delta}} G_b^T)^{-1} S_{22} - I$. The three values for each given anisotropy ratio correspond to three successive spatial refinement levels.

Results

- Analagous problem as DG convergence study, with open field lines so advection is invertible.
- Block preconditioned FGMRES, outer relative tolerance 10^{-8} .
- Inner AIR tolerance $\min\{10^{-3}, 10^{-3}/\|\mathbf{b}\|\}$, where $\mathbf{b}$ is current right-hand side.
 - Effectively absolute *and* relative tolerance.
 - If $(\Delta t \kappa_{\Delta})^{-1}$ or $\kappa_{\perp}/\kappa_{\Delta}$ is not $\ll 1$, right-hand side provided to second AIR solve can be very large, e.g., $\mathcal{O}(10^4)$. Then, relative tolerance 10^{-3} doesn't even lead to residual < 1 in norm, and outer iteration fails to converge.

Results

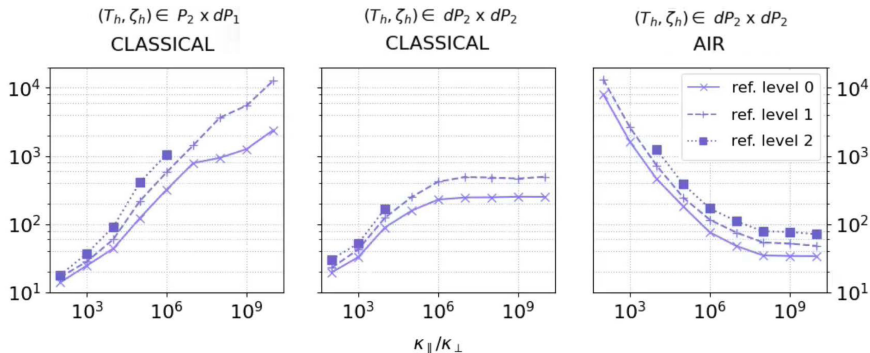


Figure: Average total inner iteration counts per time step. Left column: mixed CG, classical AMG. Center column: mixed DG, classical AMG. Right column: mixed DG, AIR.

Results

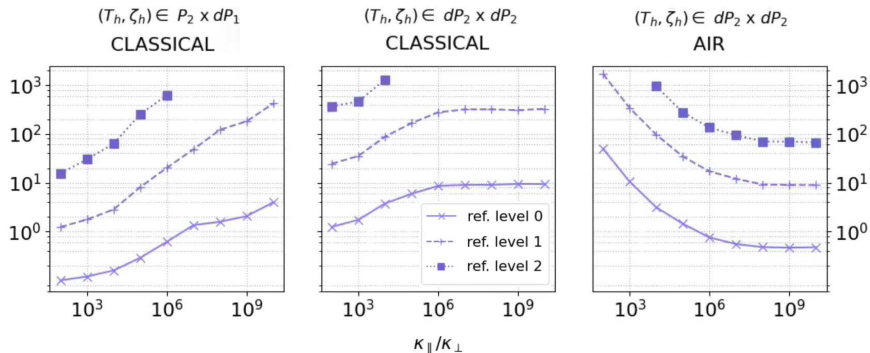


Figure: Average wall-clock times in seconds per time step. Left column: mixed CG, classical AMG. Center column: mixed DG, classical AMG. Right column: mixed DG, AIR.

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Difficulties with closed field lines

- Advection operators are no longer invertible (recall, no time derivative on advection term) $\implies$ cannot use open field lines approach directly.
- In one step, information traverses a closed field line *many* times.
- Will have closed field lines in most/all realistic problems.

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- Will have closed field lines in most/all realistic problems.

Idea: keep reordered system

$$\begin{pmatrix} -\sqrt{\kappa_{\Delta}} G_b & M \\ \frac{1}{\Delta t} M + \kappa_{\perp} L & \sqrt{\kappa_{\Delta}} G_b^T \end{pmatrix} \begin{pmatrix} T^{n+1} \\ \zeta^{n+1} \end{pmatrix} = \begin{pmatrix} F_{\zeta} \\ F_T \end{pmatrix}.$$

and apply AIR all at once to this system.

DG Tokamak test case

	AMG iters		Rel accuracy	
ref level	CG2/DG1	DG2-DG2	CG2/DG1	DG2-DG2
0	438	145	0.13	0.1
1	2263	265	0.09	0.02
2	6312	380	0.06	0.01

Table: Two toroidal planes and physical κ values, AMG iters and accuracy shown for three poloidal ref levels.

Conclusions

Review:

- Diffusion in magnetic confinement fusion is extremely anisotropic in the direction of field lines.
- Rewrote diffusion system based on directional gradients, apply discretization and solver techniques developed for advection.
- Orders of magnitude decrease in error and solve wallclock time vs. traditional/existing methods.

Conclusions

Review:

- Diffusion in magnetic confinement fusion is extremely anisotropic in the direction of field lines.
- Rewrote diffusion system based on directional gradients, apply discretization and solver techniques developed for advection.
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Next steps:

- Incorporate into larger MHD simulations.
- Better solvers for closed field lines or mixed regimes.
- Other extremely anisotropic equations??

Thank you!

Papers:

- [1] T. A. Manteuffel, S. Münzenmaier, J. Ruge, and B. S. Southworth. Nonsymmetric reduction-based algebraic multigrid.
- [2] T. A. Manteuffel, J. Ruge, and B. S. Southworth. Nonsymmetric algebraic multigrid based on local approximate ideal restriction (ℓ AIR).
- [3] B. S. Southworth, A. A. Sivas, and S. Rhebergen. On fixed-point, Krylov, and block preconditioners for nonsymmetric problems.
- [4] G. A. Wimmer, B. S. Southworth, T. J. Gregory, and X Tang. A fast algebraic multigrid solver and accurate discretization for highly anisotropic heat flux I: open field lines.
- [5] G. A. Wimmer, B. S. Southworth, and X Tang. A fast algebraic multigrid solver and accurate discretization for highly anisotropic heat flux II: closed field lines and continuous elements (*in preparation*).