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Mixed finite element formulation for solid mechanics problems

Łukasz Kaczmarczyk, Callum J. Runcie, Ananya Bijaya, Adriana Kulikova, Ross Williams, Andrei G. Shvarts, Chris J. Pearce

James Watt School of Engineering,
Glasgow Computational Engineering Centre

DOI 10.5281/zenodo.15175739

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Exponential of stretch: Neohookean

Classical Neo-Hookean potential expression:

Matrix inversion, Logarithm of Jacobian, affect robustness

$$\Psi(I_C, J) = \frac{\mu}{2}(I_C - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2$$

$$\delta\Psi = \mathbf{P} : \delta\mathbf{F}$$

$$= \{\mu(\mathbf{F} - \mathbf{F}^{-T}) + \lambda \ln J J \mathbf{F}^{-T}\} : \delta\mathbf{F}$$

New formulation exploiting logarithmic strain:

$$\Psi(I_C, J) = \mu(\exp(\mathbf{H}) - \exp(-\mathbf{H})) : \exp_{\mathbf{H}}(\delta\mathbf{H}) + \lambda\mathbf{H} : \delta\mathbf{H}$$

$$\mathbf{H} = \frac{1}{2}\left[\ln(\mathbf{F}^{\mathrm{T}} : \mathbf{F})\right], \quad \lambda^H \in \mathbb{R}$$

No matrix inversion necessary
→ numerical stability

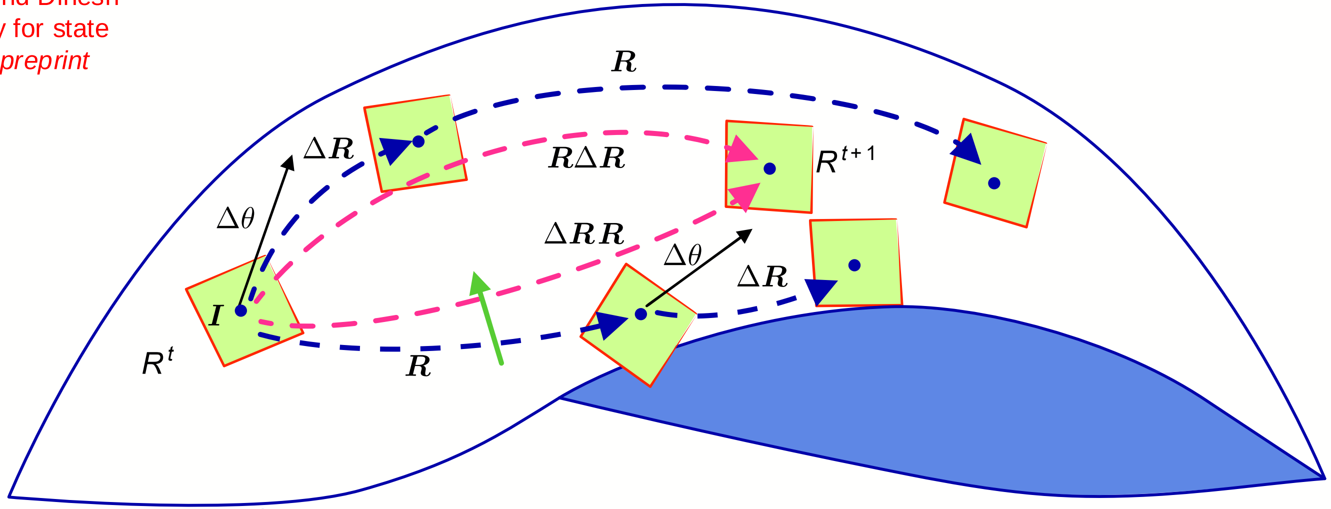
Bonet, Javier, Antonio J. Gil, and Richard D. Wood. *Nonlinear solid mechanics for finite element analysis: dynamics*. Cambridge University Press, 2021.

Poya, Roman, et al. "Generalised tangent stabilised nonlinear elasticity: An automated framework for controlling material and geometric instabilities." *Computer Methods in Applied Mechanics and Engineering* 436 (2025): 117701.

Exponential of axis of rotation: Symmetric Incremental Rotations

Sola, Joan, Jeremie Deray, and Dinesh Atchuthan. "A micro lie theory for state estimation in robotics." *arXiv preprint arXiv:1812.01537* (2018).

SO3 (Rotations in 3D)



$$(\mathbf{R}^{t+1})^T\mathbf{R}^{t+1} \approx \frac{1}{2}(\mathbf{R}\Delta\mathbf{R} + \Delta\mathbf{R}\mathbf{R})^T \cdot (\mathbf{R}\Delta\mathbf{R} + \Delta\mathbf{R}\mathbf{R}) = \mathbf{I} + \mathbf{O}^2(\Delta\boldsymbol{\theta})$$

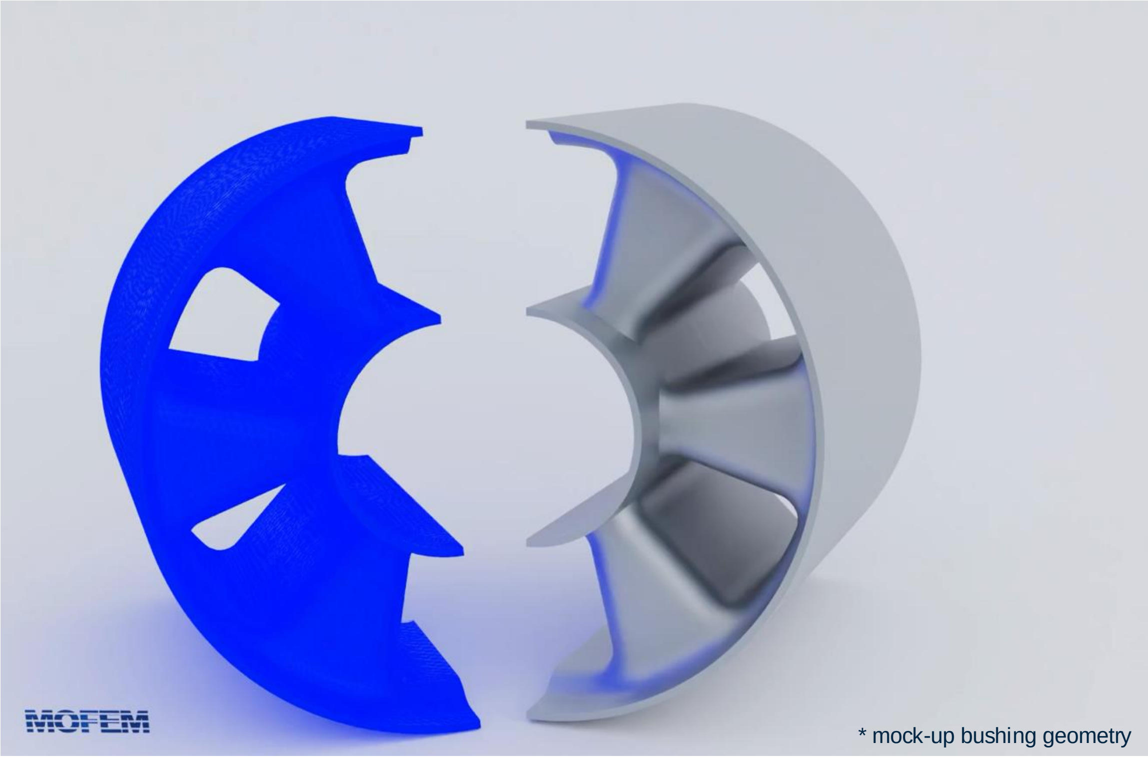
$$\Delta\mathbf{R} = \exp[\Delta\boldsymbol{\theta}]$$

Rodrigues' Rotation Formula. Rotation can be expressed with finite sum series.

$$\mathbf{F}^{t+1} = \mathbf{R}^{t+1}\exp\mathbf{H}^{t+1}$$

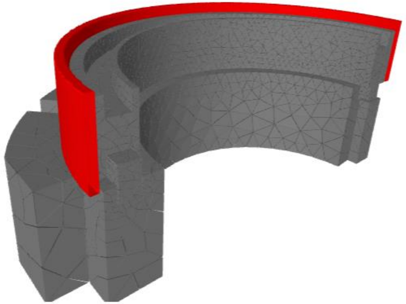
Polar decomposition (material)

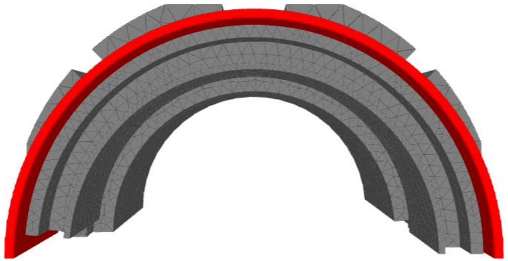
Example application: Bushing

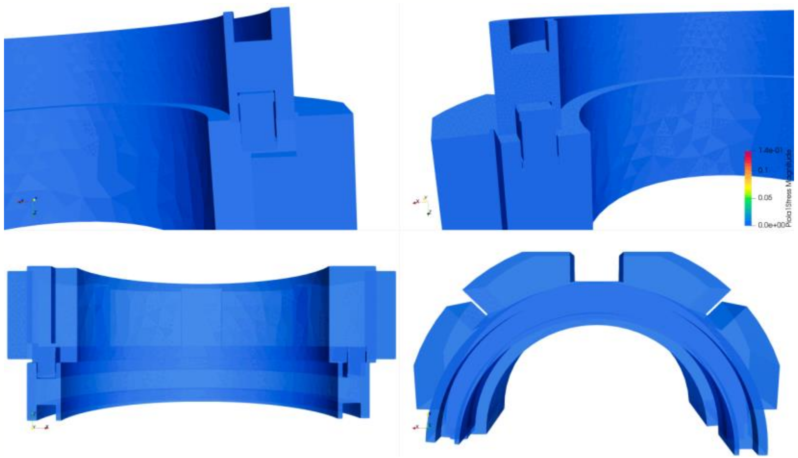


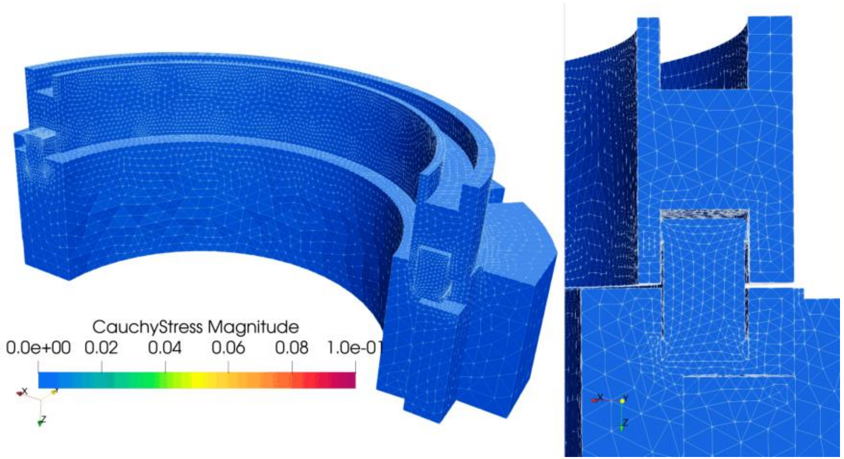
* mock-up bushing geometry

Example application: Structural integrity problems

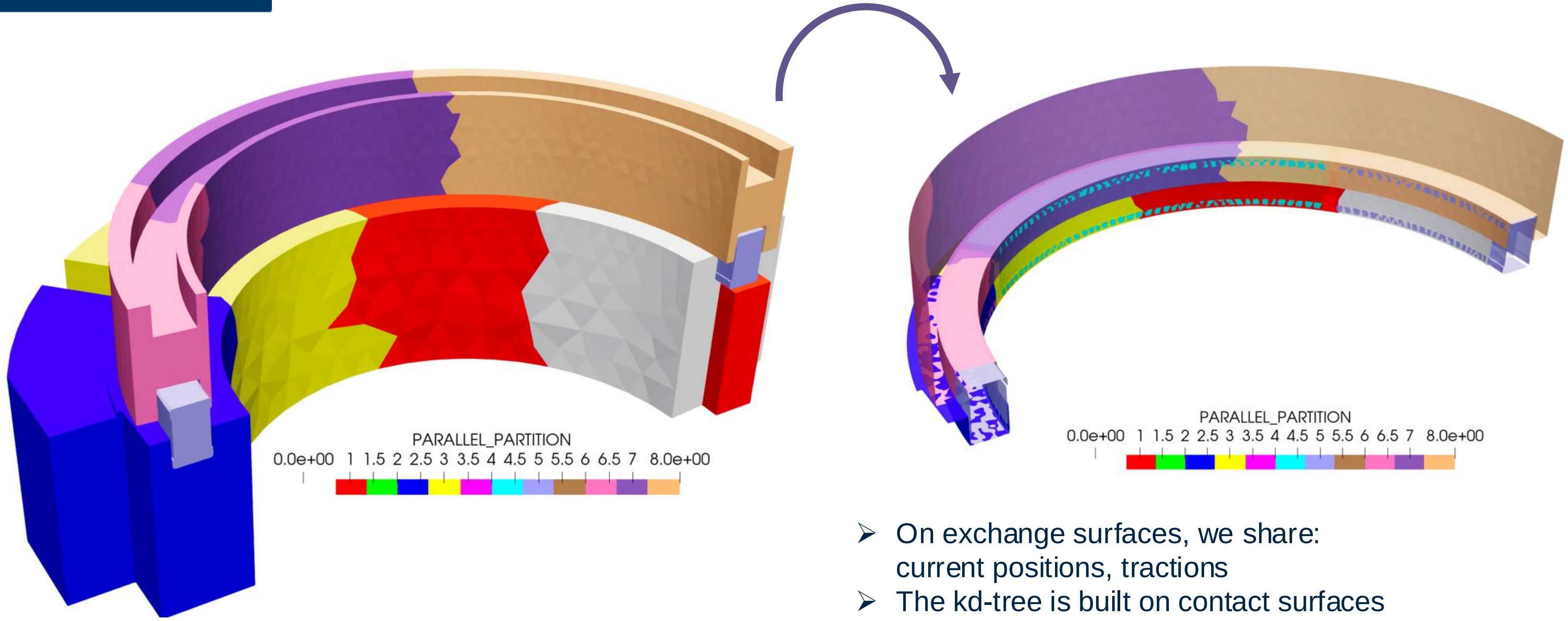








Messy issue of contact (robustness trumps efficiency in framework design)



- On exchange surfaces, we share: current positions, tractions
- The kd-tree is built on contact surfaces
- Aircraft's shadow projection method
- Contact surface mesh is arbitrarily refined (take into account HO-approximation)

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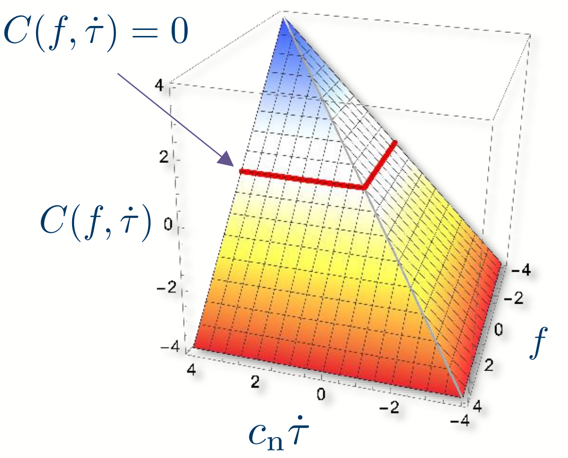
Plasticity, contact, damage & fracture

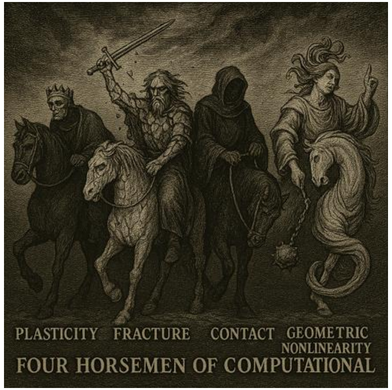


- Structural integrity problems are difficult to scale – heterogeneous materials and complex geometry with many components.

- In structural integrity problems, the devil is in the unilateral constraints, plasticity at integration points, contact on surfaces, or the crack front.

- Strongly nonlinear problems: constitutive equations, history, geometrical nonlinearities, topology evolution & boundary conditions.

- Unavoidable approximation and integration error. Plastic/Contact fronts, Cracks, Wrinkling, Creases, and Cusps, etc



Mechanics

First commit June 2013

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University of Glasgow

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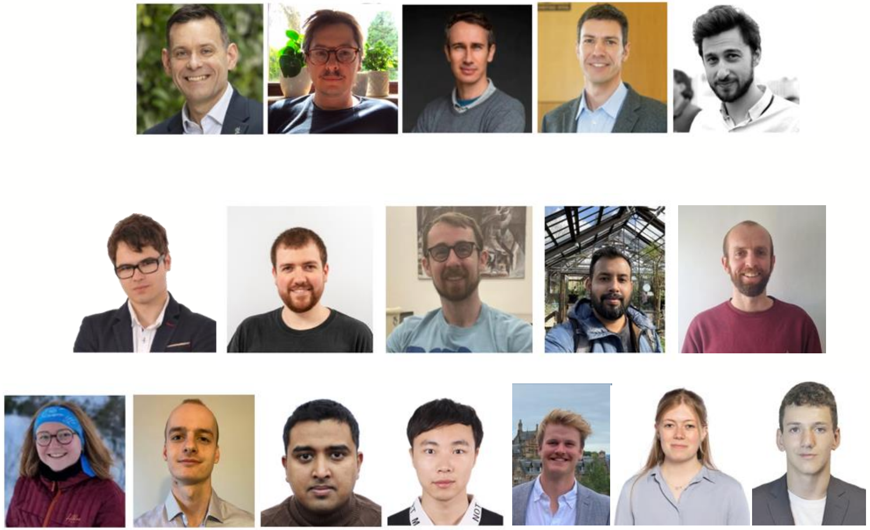
- Yingjia (Leo) Gao
- Ananya Bijaya
- Lily Sierra Fisher
- Oliver Duncan
- Bohdan Shevchenko

14 Past Contributors



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UKRI Engineering and Physical Sciences Research Council

Deep Nano

GCCC

EDF ENERGY

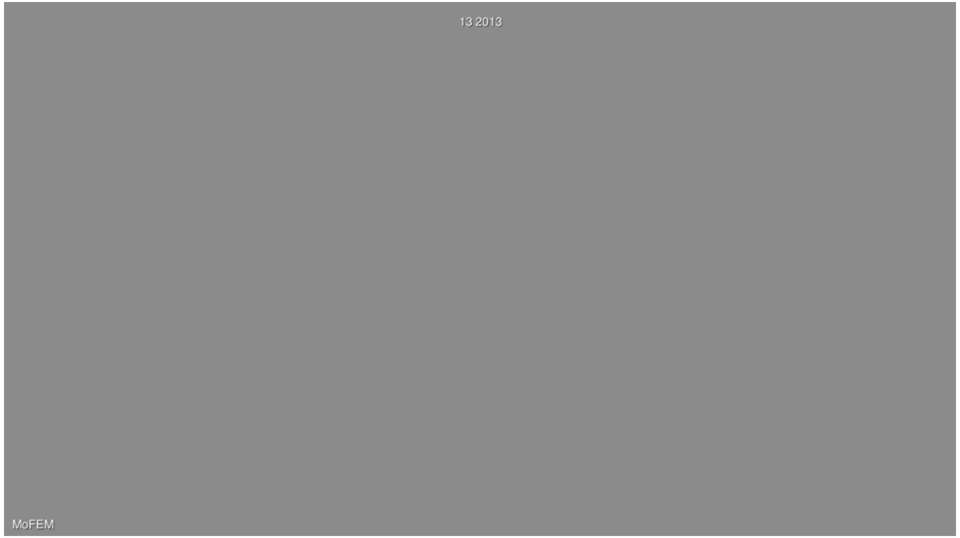
ROLLS ROYCE

MesH-Oriented Solutions

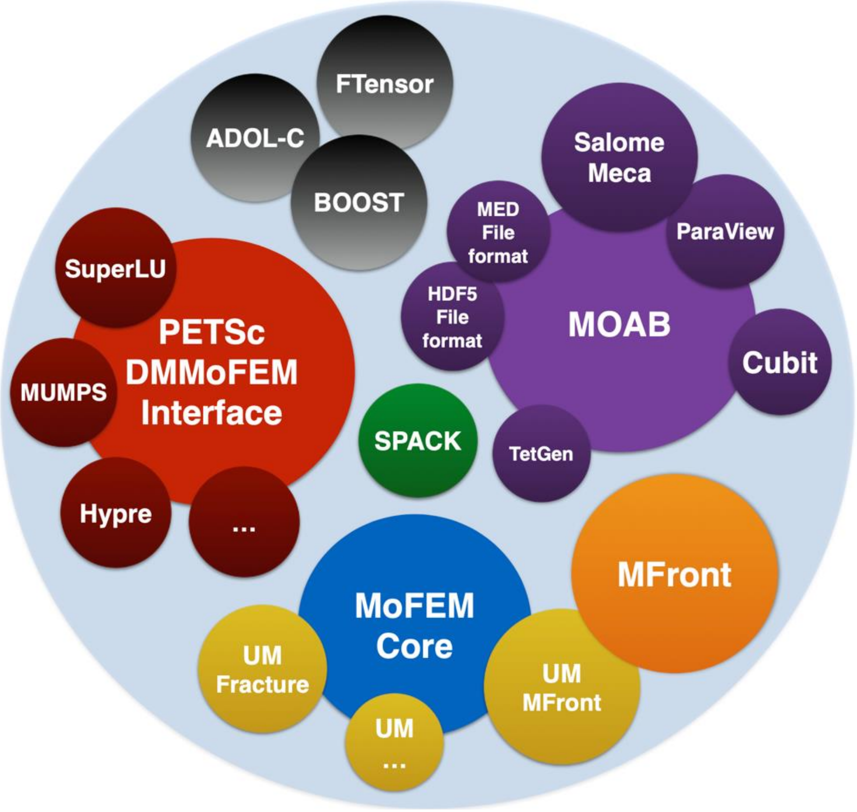
FREUDENBERG INNOVATING TOGETHER

In a nutshell

- Core library has ~270,000 lines of code
- Code made by ~50 contributors
- It would take ~70 years of work of a single programmer to write that code (according to COCOMO model - OpenHub)
- It is mostly written in C++
- Open repository and MIT license

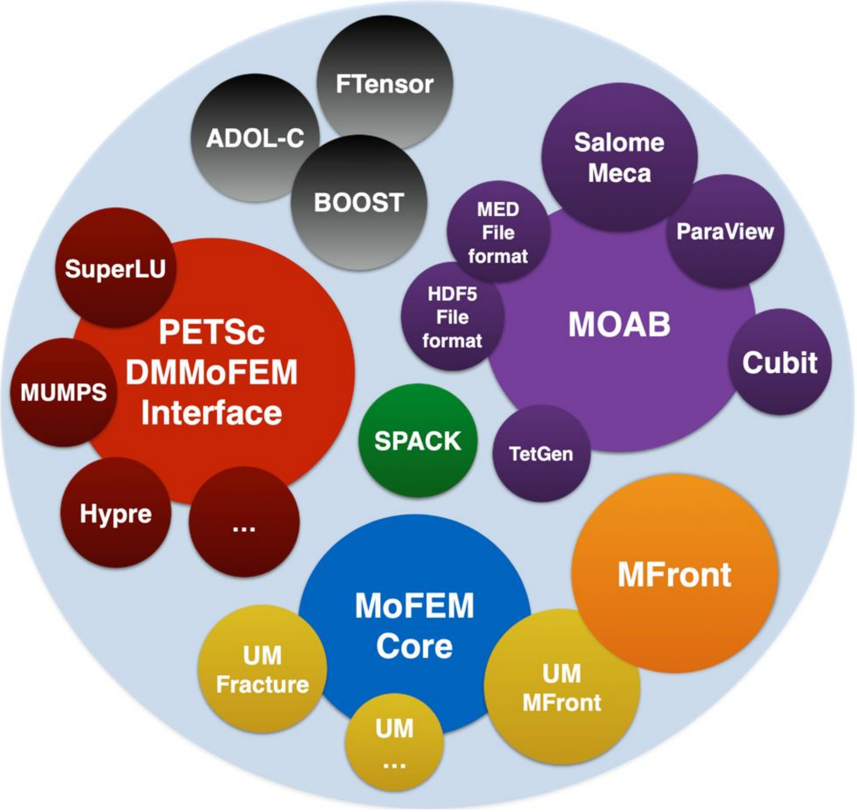


MoFEM ecosystem



MoFEM ecosystem

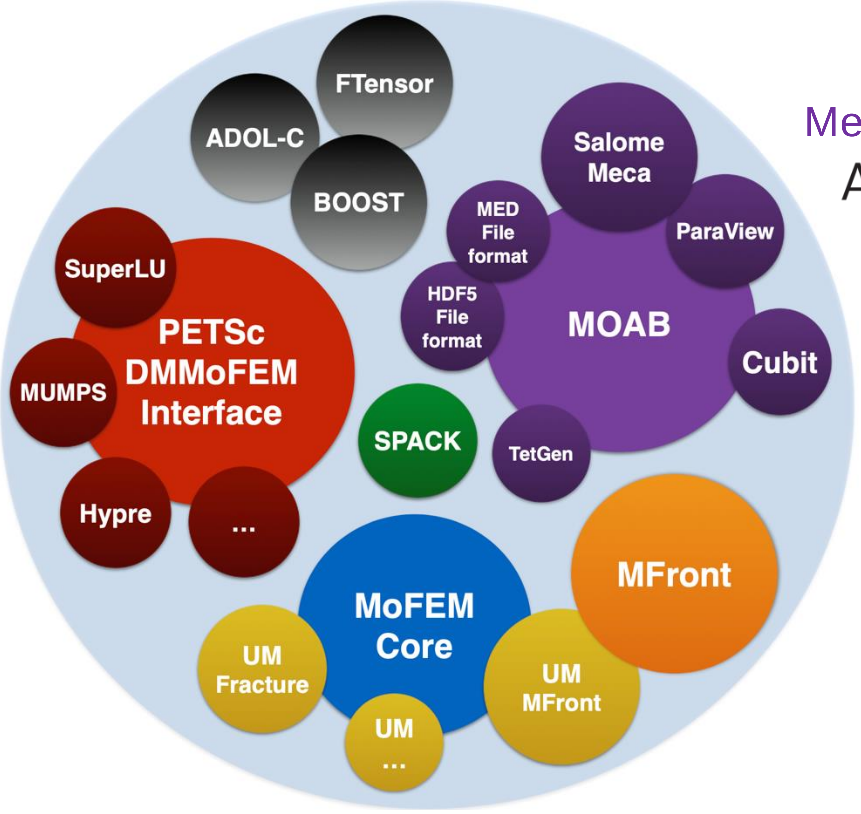
Matrices, vectors
and many solvers



MoFEM ecosystem

Mesh and data

Matrices, vectors
and many solvers



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MoFEM ecosystem

Matrices, vectors and many solvers

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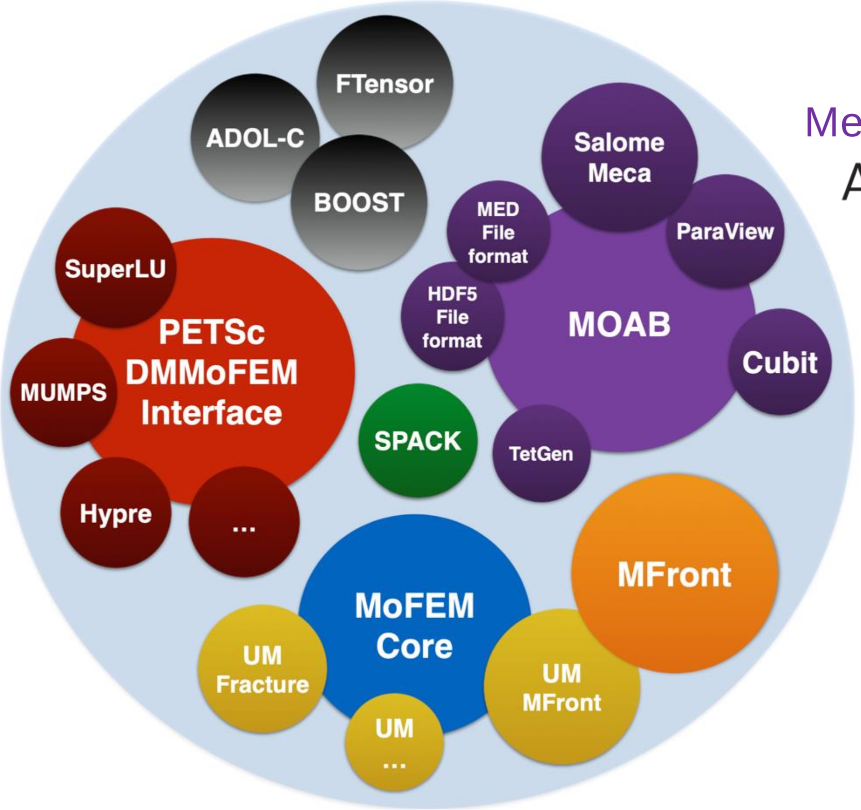

Mesh and data

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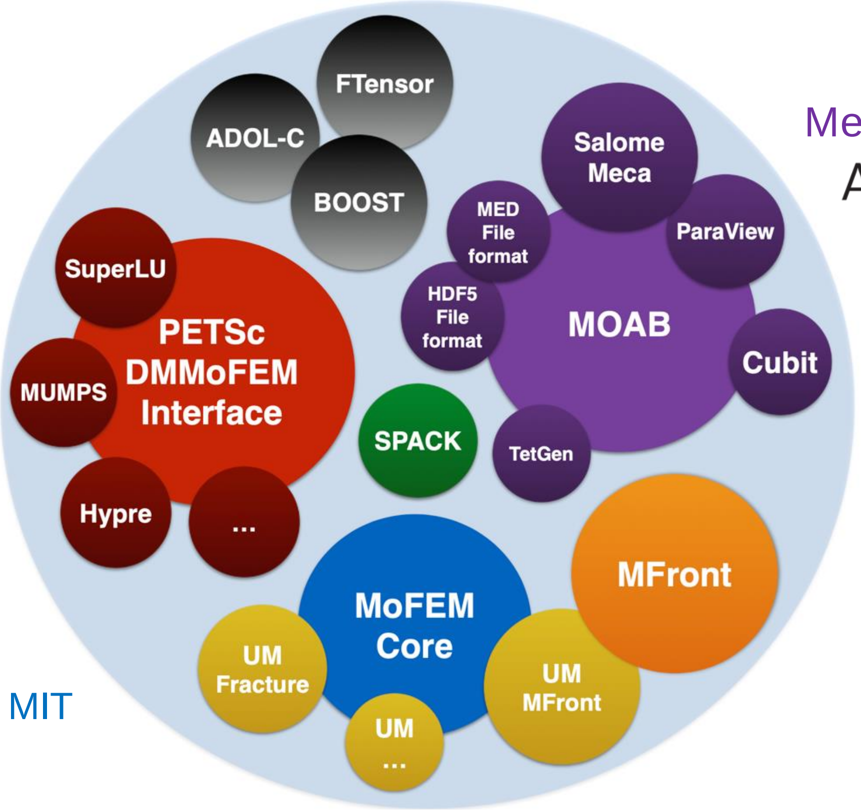
Material behaviour models

cea

MoFEM ecosystem



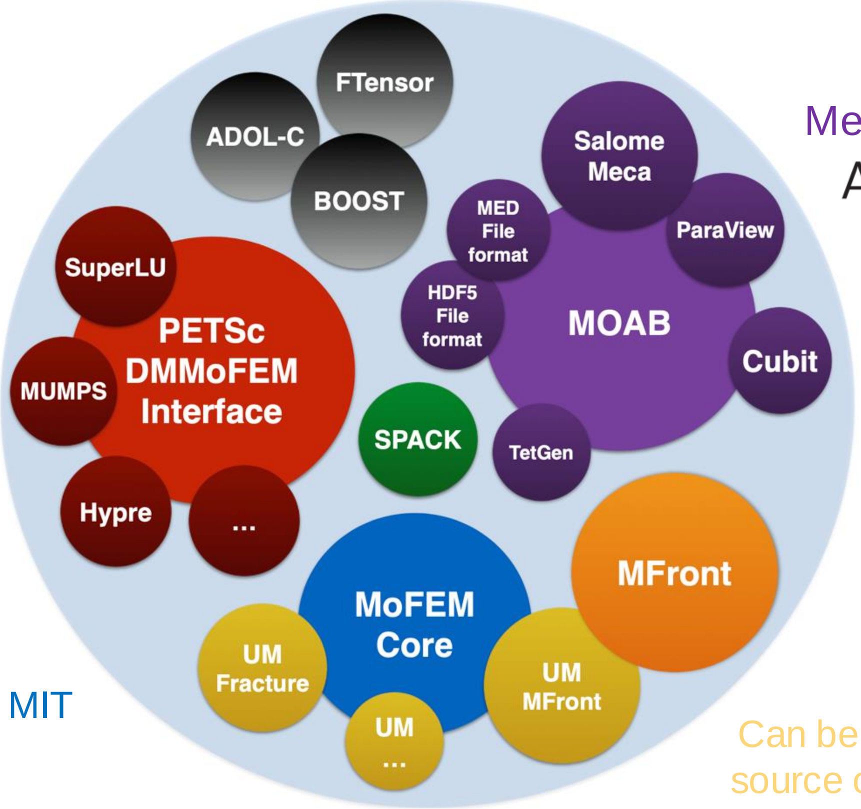
Mesh and data

Matrices, vectors and many solvers

Material behaviour models

Open-source MIT license

MoFEM ecosystem



Mesh and data

Matrices, vectors and many solvers

Material behaviour models

Open-source MIT license

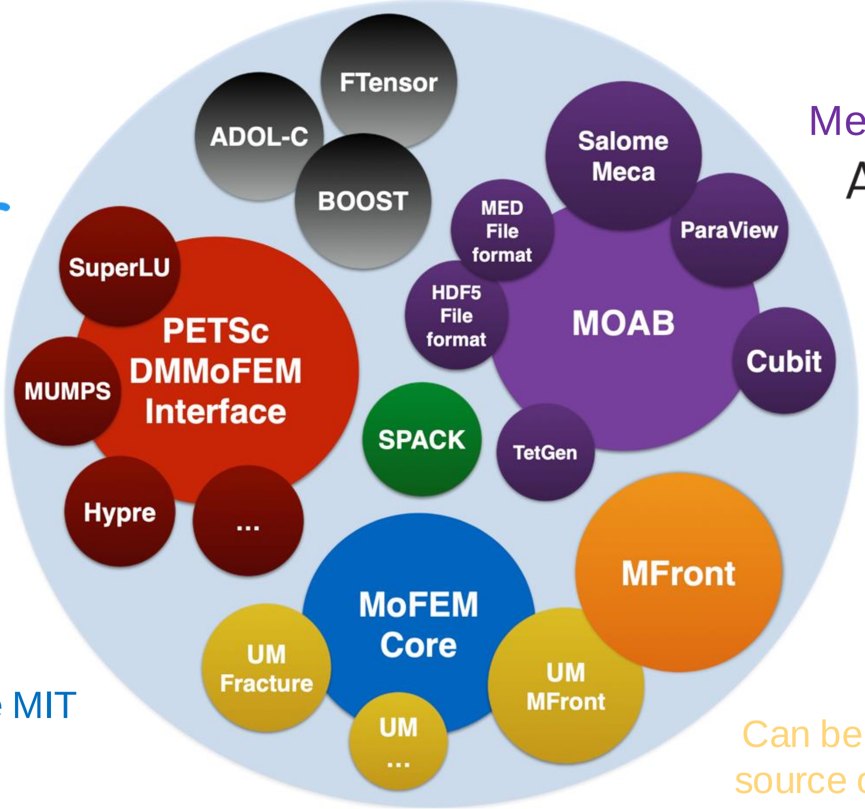
Can be either open-source or proprietary

MoFEM ecosystem

All can be shipped in a Docker container

Matrices, vectors and many solvers

Open-source MIT license

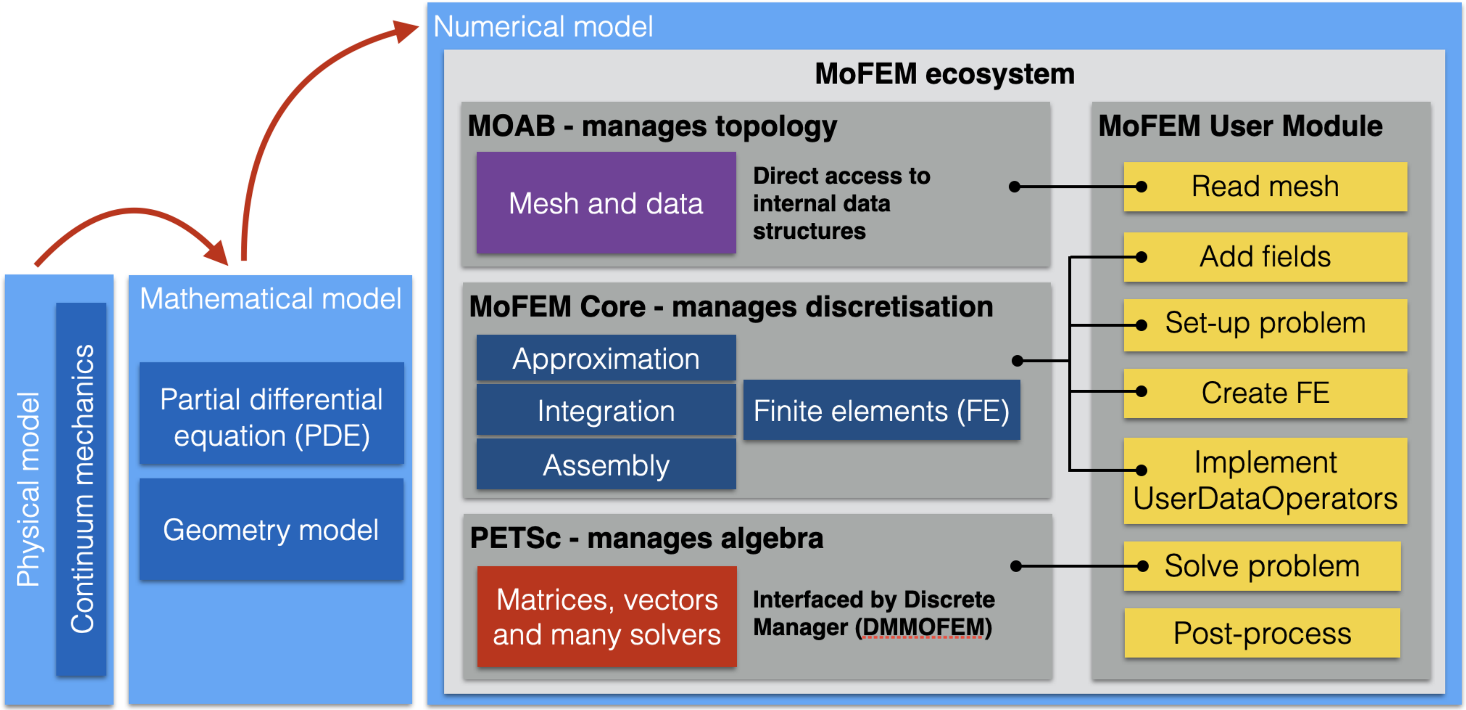


Mesh and data

Material behaviour models

Can be either open-source or proprietary

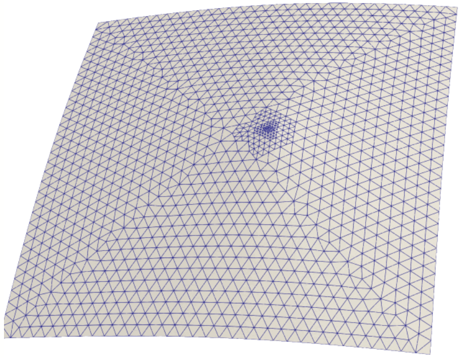
High Level Design



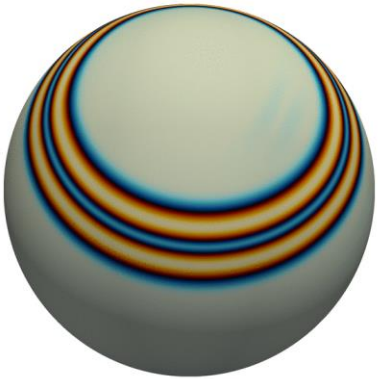
MoFEM Features

- We aim at the whole de Rham complex: L2, H-div, H-curl, and H1.
- L2/DG, H1/DG, and other energetic spaces are also included.
- Hierarchical approximation bases.
- Scalar, vectorial and tensorial bases.
- Hierarchical (p-adaptivity) and Beristain-Bezier base

DG-upwind advection of level set: Tutorial ADV-3.



Shallow-Wave equation on generic curved surfaces



MoFEM Design – Industry First Approach

- Core library (abstraction levels) and modules with physics implementation – two different repositories, licensing, copyright when needed.
- Hollywood model (you do not call us, we call you) – development patterns - high code coverage - research code short pathway to application.

```
MoFEM::Core core(moab);
MoFEM::Interface& m_field = core;
// define fields
CHKERR m_field.add_field("DISP",H1,AINSWORTH_LEGENDRE_BASE,3);
CHKERR m_field.add_field("FLUX",HDIV,DEMKOWICZ_JACOBI_BASE,1);
// meshset consisting all entities in mesh
EntityHandle root_set = moab.get_root_set();
// add entities to field
CHKERR m_field.add_ents_to_field_by_TETs(root_set,"DISP");
CHKERR m_field.add_ents_to_field_by_TETs(root_set,"FLUX");
// set app. order (that is boring same order to all)
int order = 5;
CHKERR m_field.set_field_order(root_set,MBTET,"DISP",order);
CHKERR m_field.set_field_order(root_set,MBTET,"FLUX",order);
// build
CHKERR m_field.build_fields();
```

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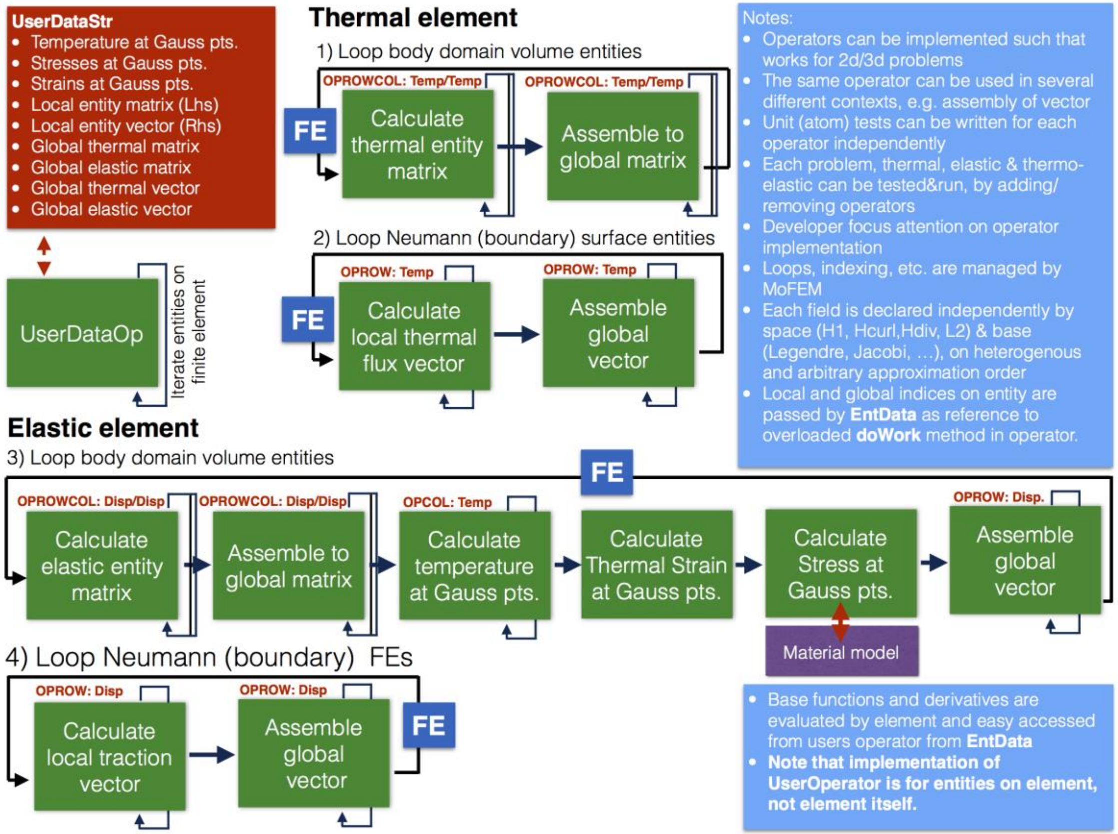
Finite Element is a pipeline of operators



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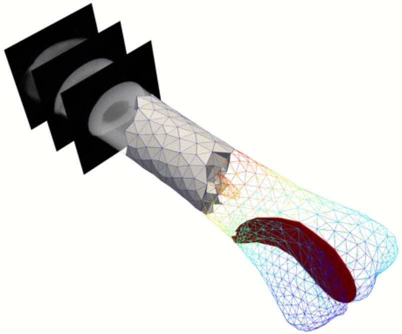
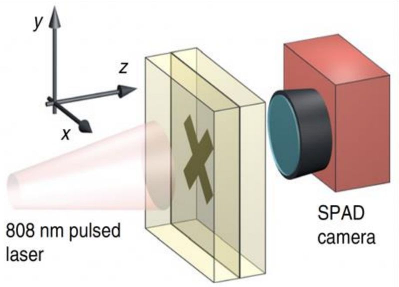
Leverage testing through science applications

Bone remodelling & fracture

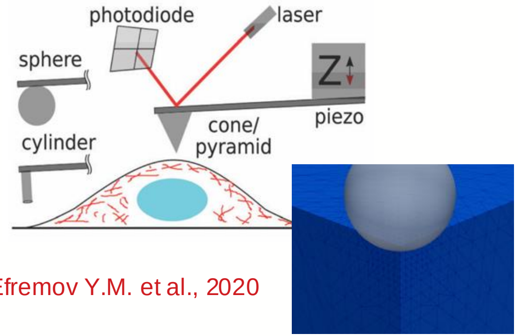


Lewandowski K., et al., 2021

Photon diffusion through turbid media



Simulation of nanoindentation for AFM of cells



Efremov Y.M. et al., 2020

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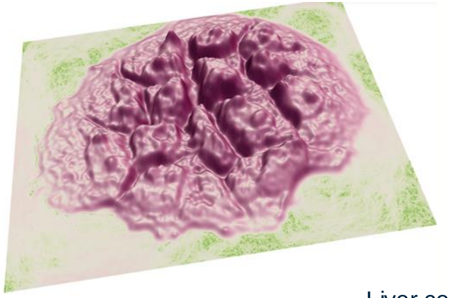
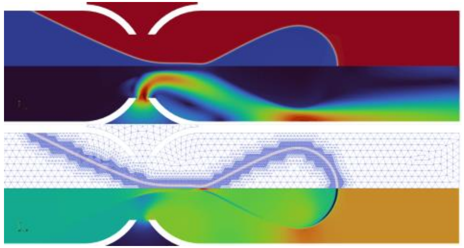
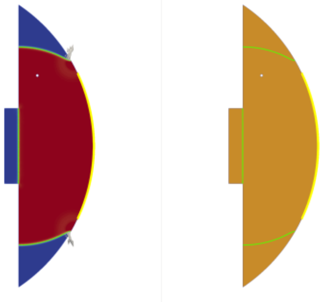


Liver cell

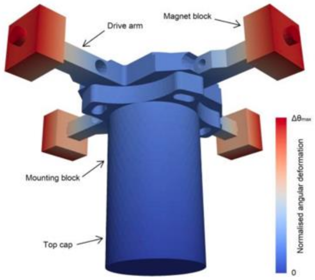
Microfluidics





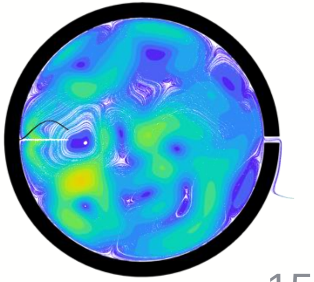
D. Lockington, et al. 2022

Geotechnics



Rieman L., et al. 2023

Fluid-structure interaction



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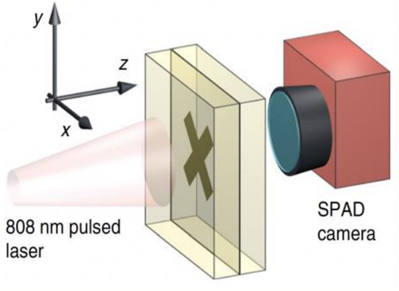

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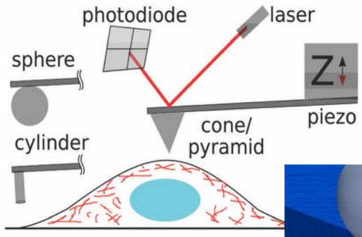

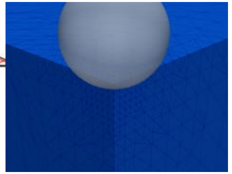

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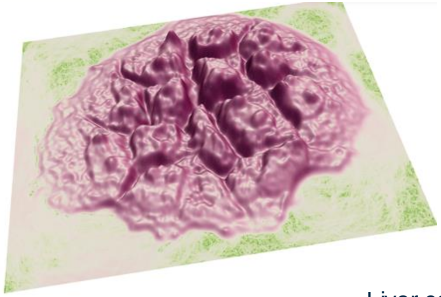

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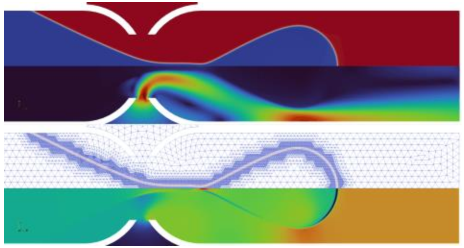
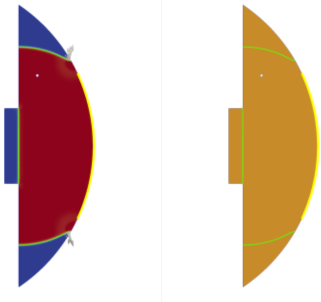
Microfluidics





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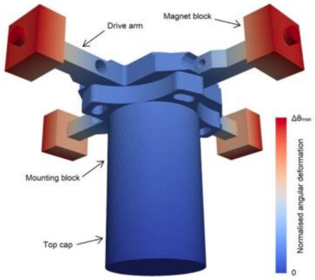

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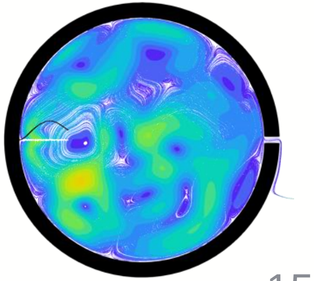

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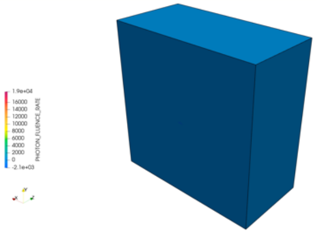

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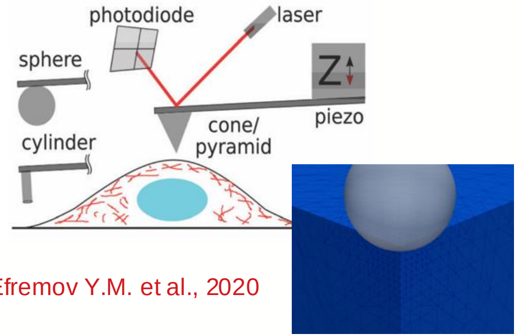

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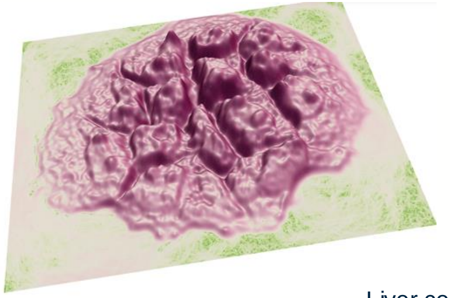

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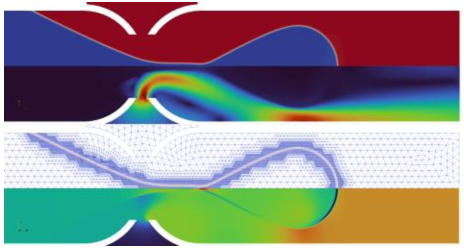

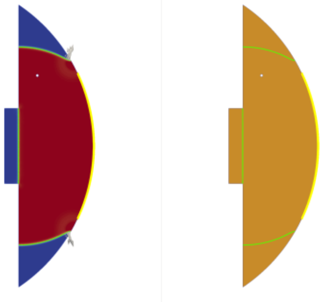

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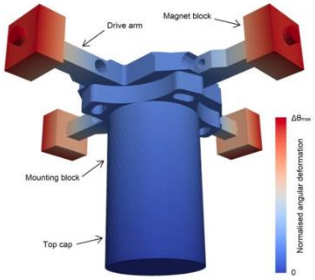

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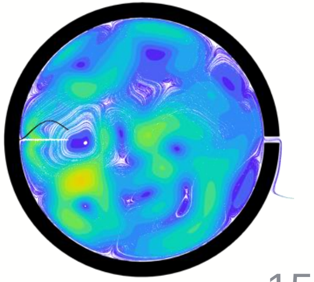
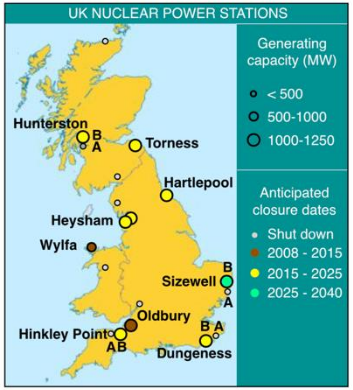


Nuclear power in the UK

Advanced gas-cooled reactors (AGR)

Nuclear power in the UK

Advanced gas-cooled reactors (AGR)



≈20% of energy
from nuclear power

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Nuclear power in the UK

Advanced gas-cooled reactors (AGR)

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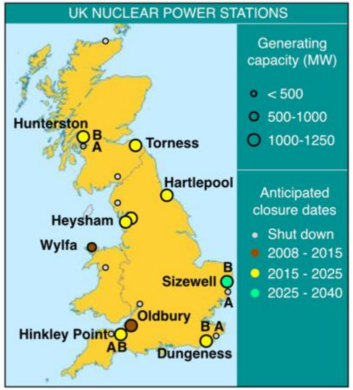

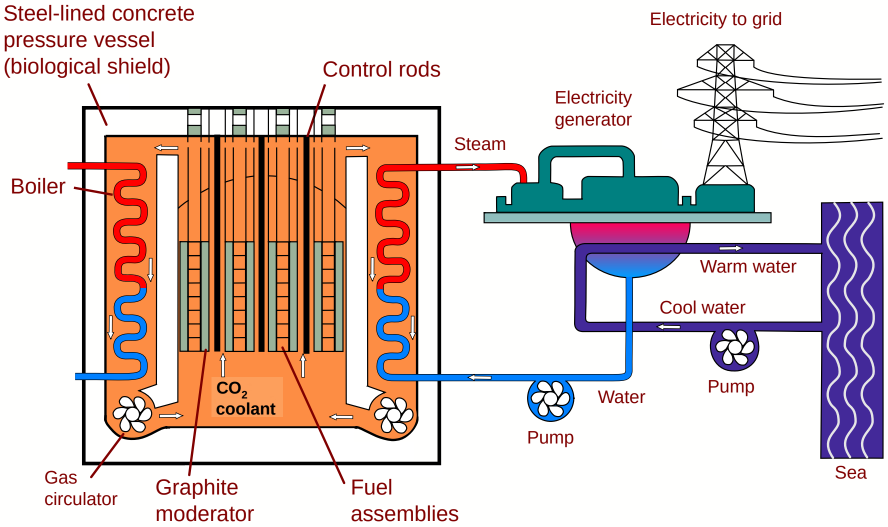

≈20% of energy from nuclear power

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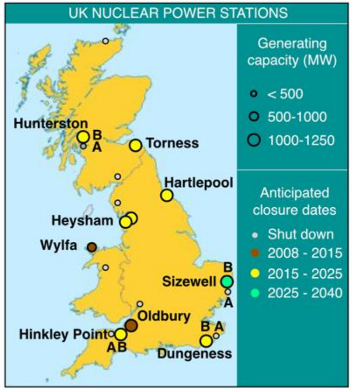
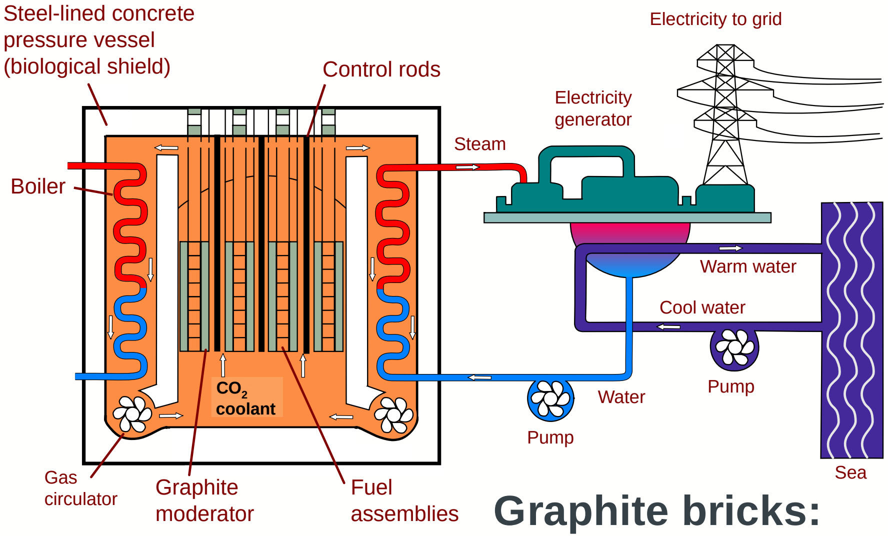
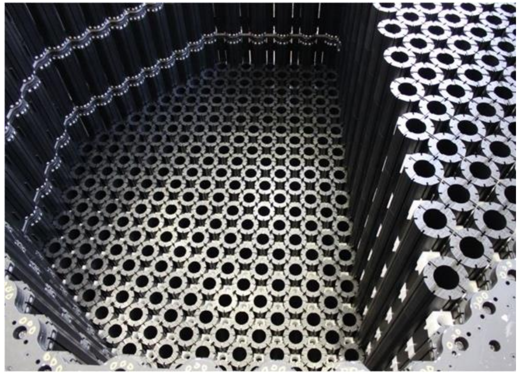






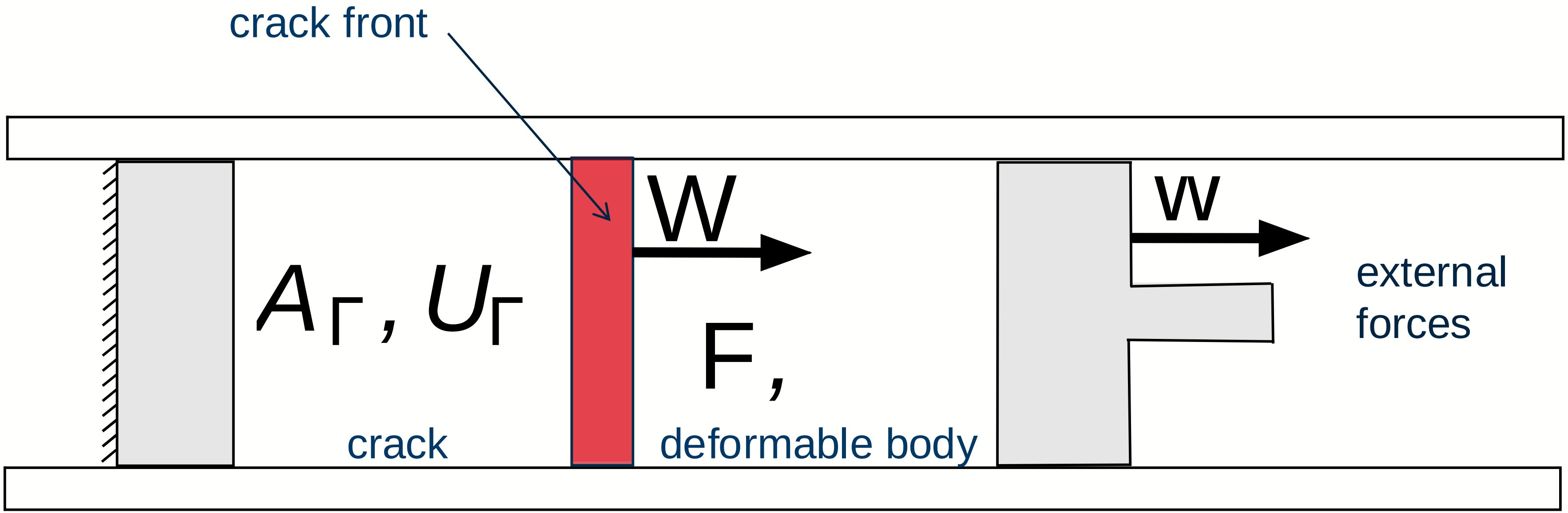
≈20% of energy from nuclear power

Graphite bricks:

- neutron moderator
- mechanical stability
- thermal inertia

Thermomechanical model

First Law

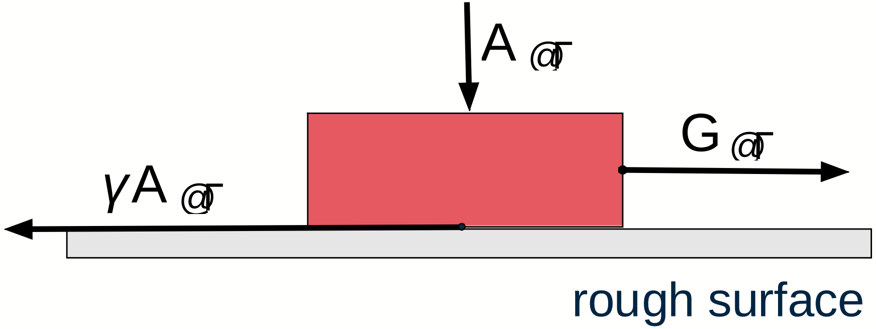


Fracture process

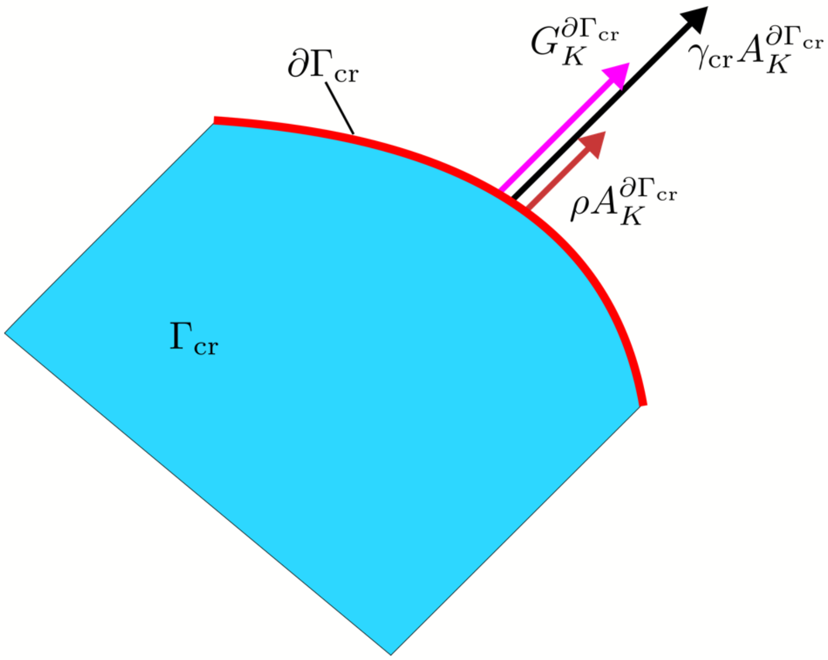
Second Law

Entropy (nothing) is never like it was before

$$D := \gamma \dot{W} \cdot A_{\partial\Gamma} \geq 0$$



Force balance at the crack front



Balance:

$$G_K^{\partial\Gamma_{\rm cr}} - (g_{\rm cr}/2 - \rho) A_K^{\partial\Gamma_{\rm cr}} = 0$$

3 cases:

(i) body is not loaded $\Leftrightarrow G_K^{\partial\Gamma_{\rm cr}} = 0 \Leftrightarrow \rho = g_{\rm cr}/2$

(ii) crack propagates $\Leftrightarrow \rho = 0$

(iii) intermediate state $\Leftrightarrow \rho > 0$ and $G_K^{\partial\Gamma_{\rm cr}} \neq 0$

KKT for crack propagation:

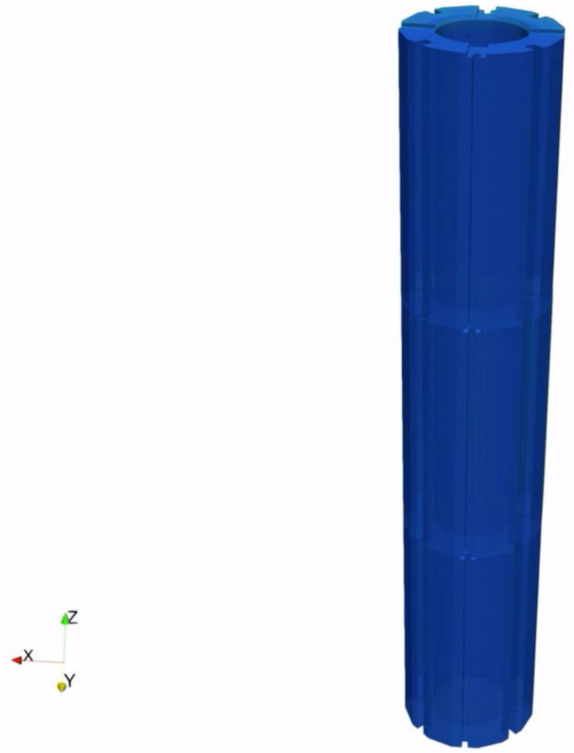
$$A_K^{\partial\Gamma_{\rm cr}} \dot{W}_K \geq 0, \qquad \rho \geq 0, \qquad \rho A_K^{\partial\Gamma_{\rm cr}} \dot{W}_K = 0$$

Complementarity function:

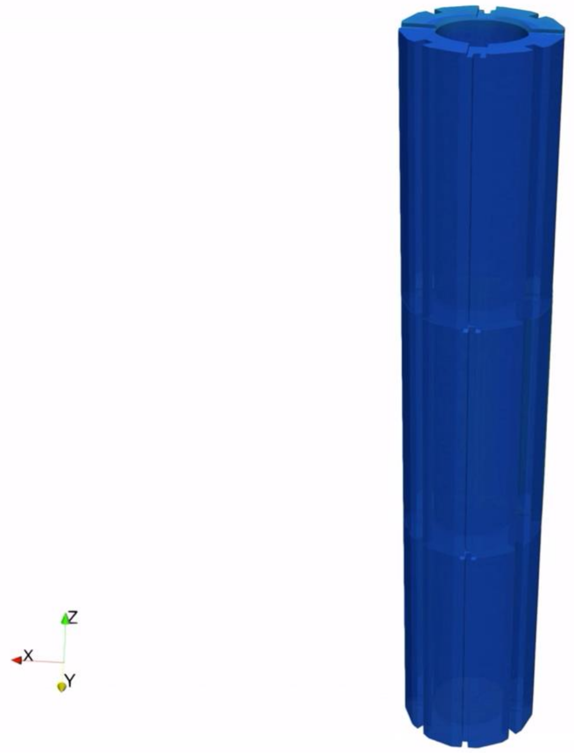
$$C_{\rm cr}(\rho, \dot{W}_K) := \rho - \max\left(0, \rho - c_{\rm cr} A_K^{\partial\Gamma_{\rm cr}} \dot{W}_K\right)$$

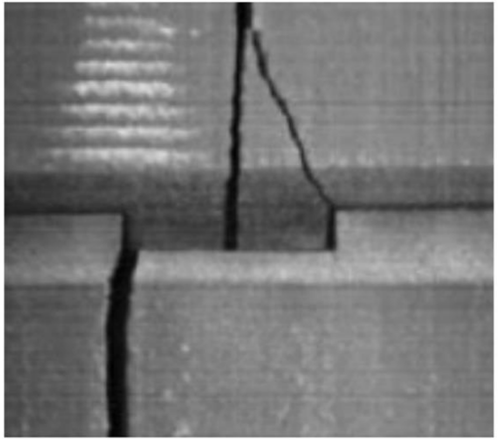
Fracture of irradiated graphite bricks

Fracture of irradiated graphite bricks



Fracture of irradiated graphite bricks



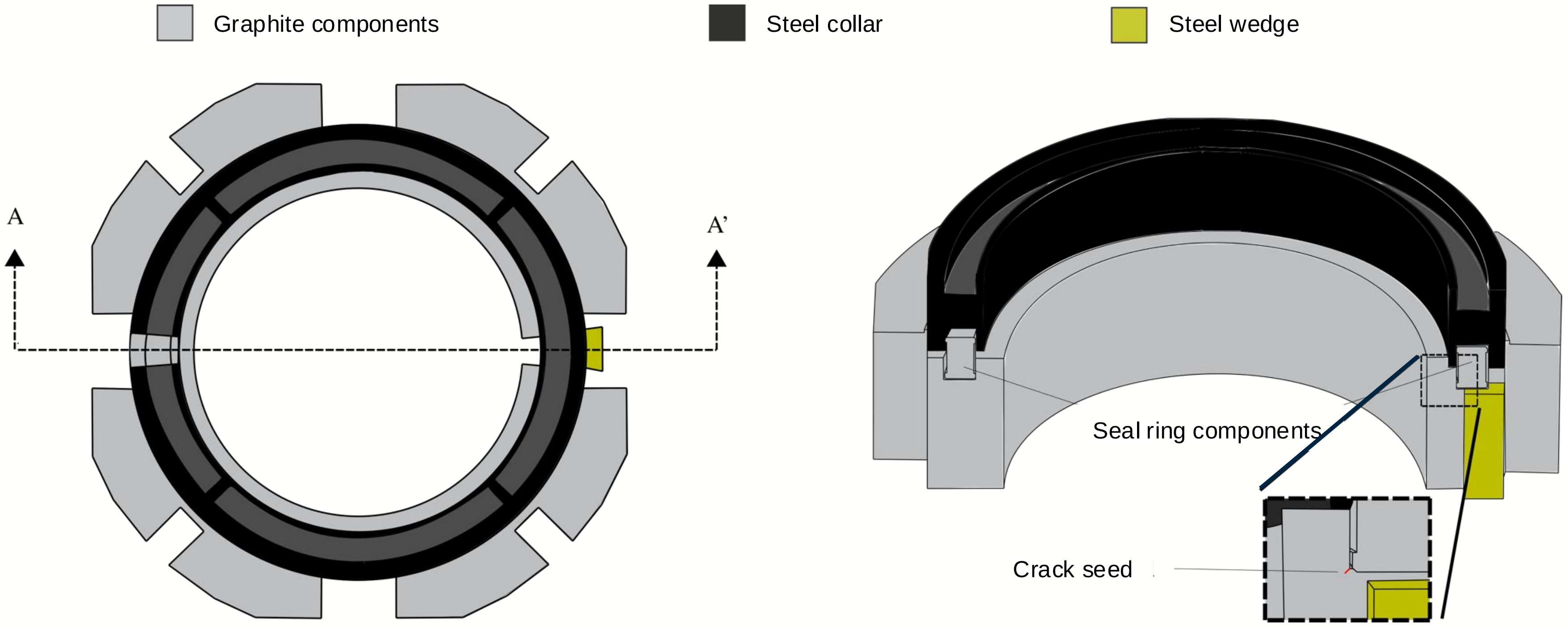


EDF ENERGY

Athanasiadis, I., Shvarts, A.G. et al. *CMAME* (2023)

Scan to access paper

Analysis geometry



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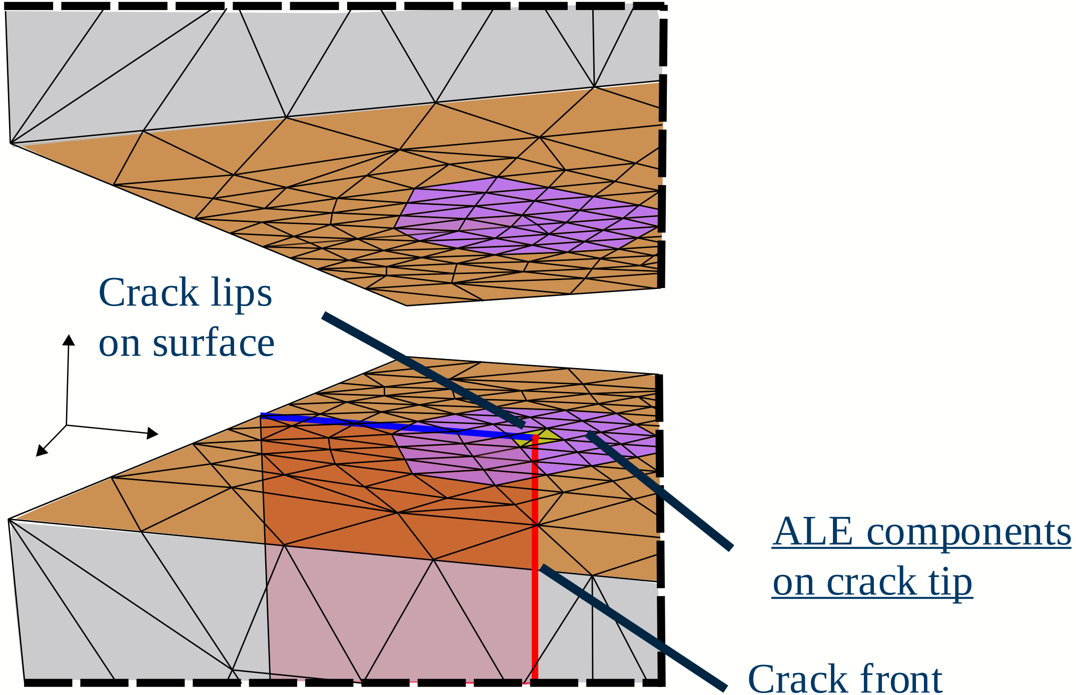
Simulation



MOFEM
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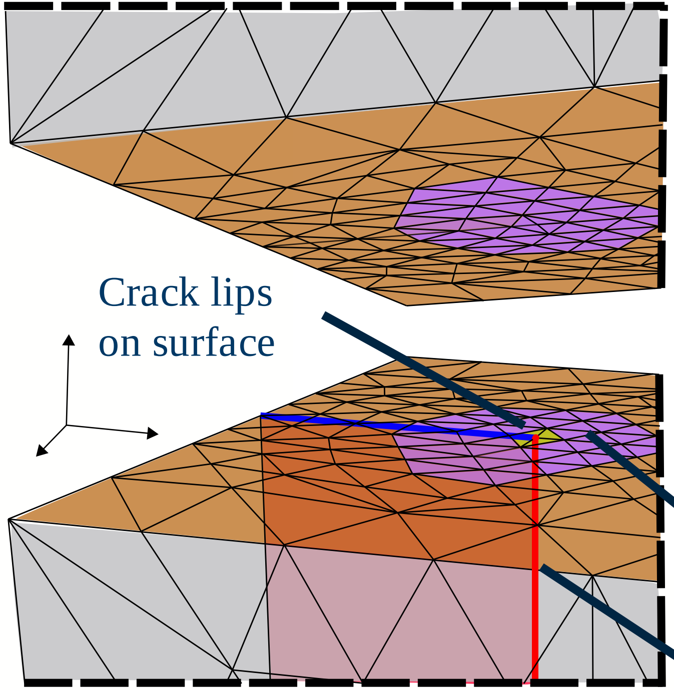
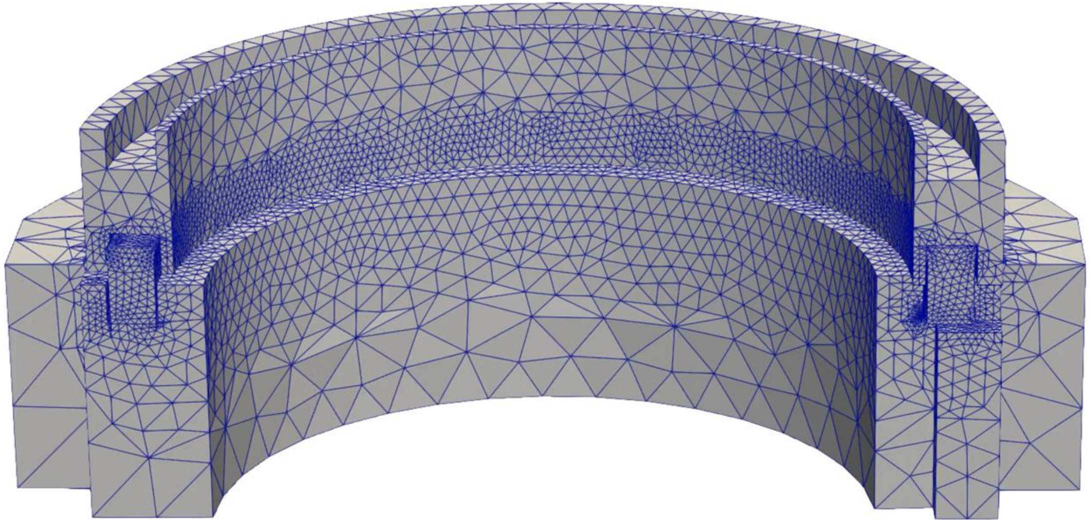
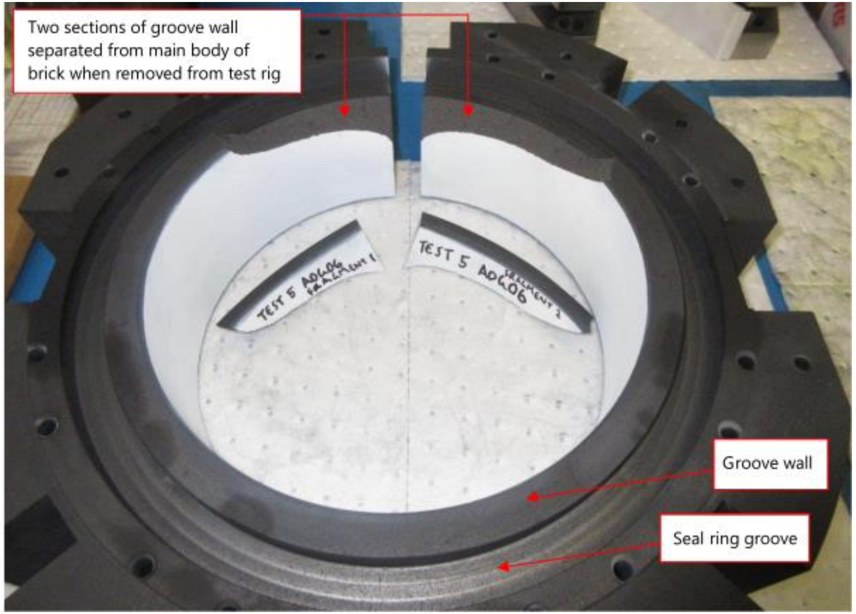
Simulation

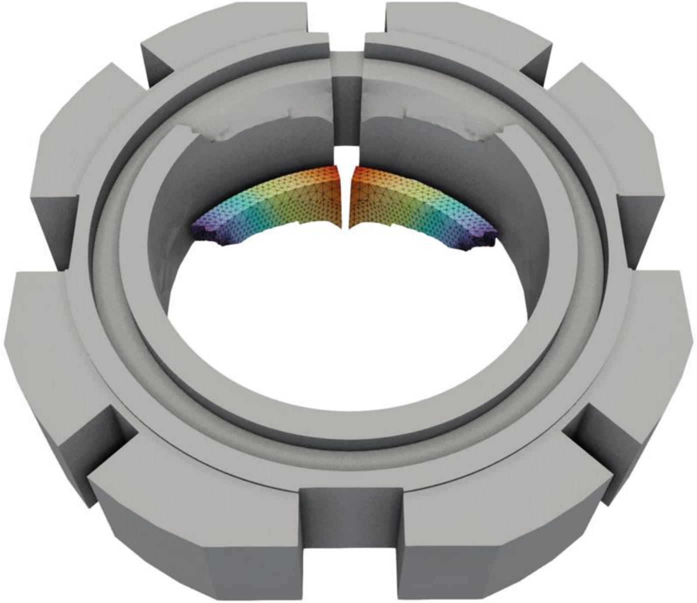




Comparison with the experiment



Jacobs



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Mixed Elasticity Formulation

Generalised Hu-Washizu functional $\mathbf{U} = \ln\left(\mathbf{H}\right)$

$$\Pi = W(\boldsymbol{U}) + \int_{\Omega_0} \left(\frac{\partial x_i}{\partial X_J} - R_{iK} U_{KJ} \right) P_{iJ} \mathrm{d}V - \int_{\Omega_0} u_i f_i \mathrm{d}V - \int_{\gamma_0} \bar{u}_i t_i \mathrm{d}S$$

Consistency equation:

$$\frac{\partial \Pi}{\partial P_{iJ}} \delta P_{iJ} = \int_{\Omega_0} \frac{\partial x_i}{\partial X_J} \delta P_{iJ} \mathrm{d}V - \int_{\Omega_0} R_{iK} U_{KJ} \delta P_{iJ} \mathrm{d}V$$

Physical equation:

$$\frac{\partial \Pi}{\partial U_{KL}} \delta U_{KL} = \frac{\partial W}{\partial U_{MN}} \partial U_{MN} - \int_{\Omega_0} R_{iK} U_{KL,MN} \delta U_{MN} P_{iL} \mathrm{d}V$$

Conservation of angular momentum:

$$\frac{\partial \Pi}{\partial \theta_\alpha} \delta \theta_\alpha = -\frac{1}{2} \int_{\Omega_0} \left(\delta\theta_\alpha \varepsilon_{ij\alpha} R_{jK} + R_{iN} \varepsilon_{NK\alpha} \delta\theta_\alpha \right) U_{KL} P_{iL} \mathrm{d}V$$

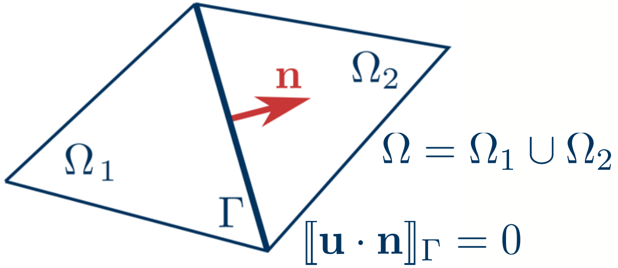
Conservation of linear momentum:

$$\frac{\partial \Pi}{\partial u_i} \delta u_i = \int_{\Omega_0} P_{iJ} \frac{\partial \delta u_i}{\partial X_J} \mathrm{d}V - \int_{\Omega_0} \delta u_i f_i \mathrm{d}V$$

H(div) main properties

$$H(\mathrm{div};\Omega) := \left\{ \mathbf{u} \in \left[L^2(\Omega)\right]^2 \mid \mathrm{div}\, \mathbf{u} \in L^2(\Omega) \right\}$$

Normal component of H(div) function is continuous across any inner boundary:



Natural space for flow/diffusion problems

(Classic example: *Raviart-Thomas* FE space)

[1] Boffi D, Brezzi F, Fortin M. Mixed finite element methods and applications (2013)

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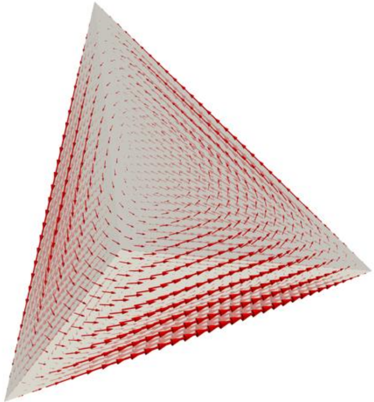
Spaces

$$\delta\mathbf{P} \in U_0^h \subset H_0^{\mathrm{div}}(\Omega_0^h) := \{\delta\mathbf{P} \in H^{\mathrm{div}}(\Omega_0^h) : N_j \delta P_{ij} = 0 \text{ on } \partial\Omega_0^{h,u}\}$$
$$\tilde{\mathbf{P}} + \mathbf{P} \in U^h := \{\tilde{\mathbf{P}} + \mathbf{P} : \mathbf{P} \in H_0^{\mathrm{div}}(\Omega_0^h)\}$$
$$\mathbf{u}, \boldsymbol{\omega}, \delta\mathbf{u}, \delta\boldsymbol{\omega} \in C^h \subset L^2(\Omega_0^h)$$
$$\mathbf{H}, \delta\mathbf{H} \in S^h \subset S := \{H_{ij} \in L^2(\Omega_0^h) : H_{ij} = H_{ij}^T\}$$



Riviart-Thomas (RT) ← Demkowicz recipe

$$H^{\mathrm{div}}(\mathcal{B}_0^h) = \mathcal{RT}^k(\mathcal{K}) + \mathrm{curl}((\mathrm{curl}\tilde{\mathbf{A}}^k(\mathcal{K}))\mathbf{b}_{\mathrm{K}})$$

Antisymmetric matrix homogenous polynomial

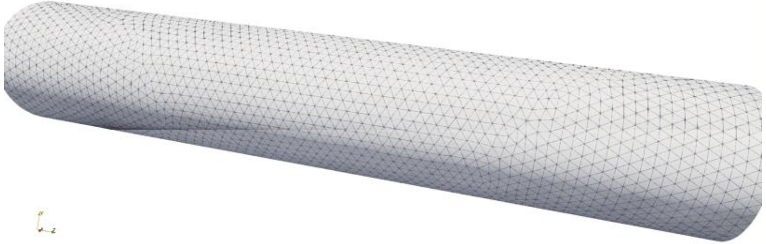
matrix bubble

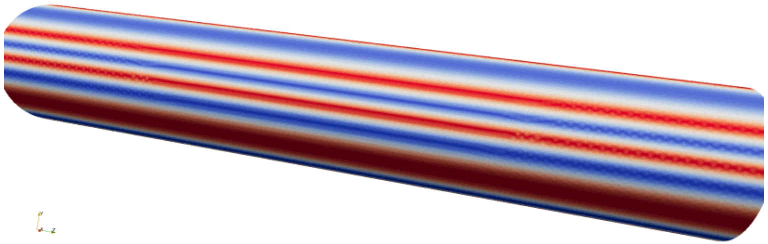
$$b_{\mathrm{K}ij} = \sum_{l=0}^{2} \lambda_{l-3}\lambda_{l-2}\lambda_{l-3}\lambda_{l,i}\lambda_{l,j}$$

Gopalakrishnan, Jayadeep, and Johnny Guzmn. "A second elasticity element using the matrix bubble." IMA Journal of Numerical Analysis 32.1 (2012): 352-372.

Why mixed formulation with stress approximation

- ✓ Separation of non-linearities as different equations
- ✓ Conservation equations (momentum flux continuity) is satisfied a priori
- ✓ Trades floating point operations to local and temporal memory access
- ✓ Weaker (ultra weak formulation) suitable for unilateral constraints emerging from contact, fracture, plasticity and geometric instabilities
- ✓ Sparse Dense Block Structure
- ✓ Future hardware ready (GPUs)





Updated Lagrangian formulation

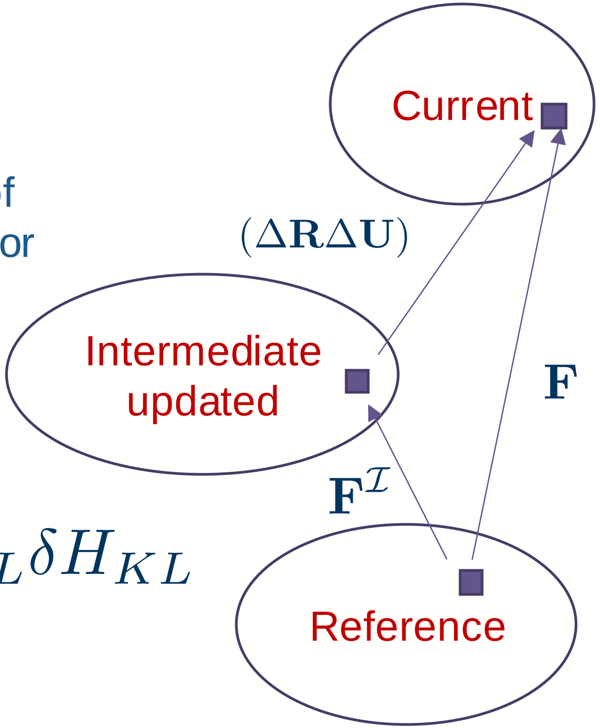
$$F_{iJ} = R_{iK} U_{KJ} = R_{iK} [\mathrm{Exp}(\mathbf{H})]_{KJ}$$

Deformation gradient

Orthonormal rotation tensor

Symmetric stretch tensor

Logarithm of stretch tensor



Physical values of stretches are larger than 1, while logarithm of stretch can take any finite value.

$$U_{IJ} = \mathrm{Exp}(\mathbf{H})_{IJ}$$

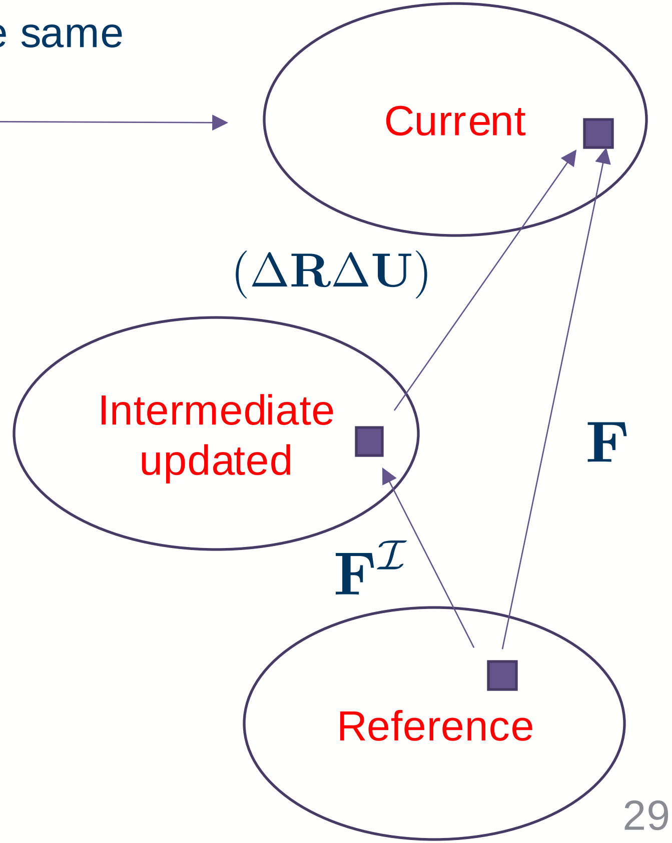
Variation: $\delta U_{IJ} = \mathrm{Exp}(\mathbf{H})_{IJ,KL} \delta H_{KL}$

Mathematical formulations for small strains:
Gopalakrishnan, J., & Guzmán, J. (2012). A second elasticity element using the matrix bubble. *IMA Journal of Numerical Analysis*, *32*(1), 352-372.

Total vs update Lagrangian formulation

Objectivity: Lagrangian and Total formulations have to yield the same result. Not exactly the case for mixed formulation. ???

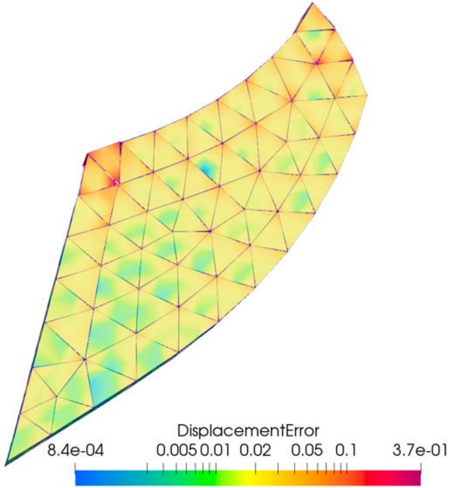
$$\|\mathbf{u} - \mathbf{u}^{h,k}\|_{L^2(\Omega)} \leq C h^{k+1} |\mathbf{u}|_{H^{k+1}(\Omega)}$$



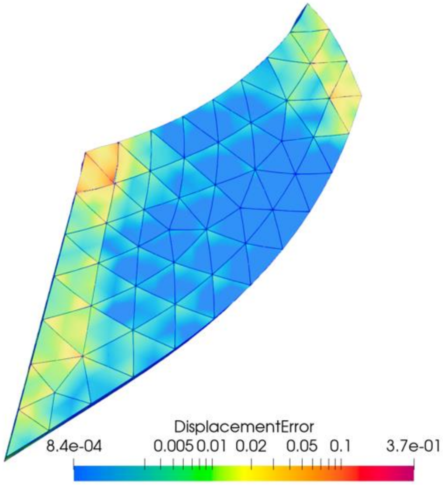
Stenberg, R. (1988). A family of mixed finite elements for the elasticity problem. Numerische Mathematik, 53, 513-538.

Cook beam (p-refinement)

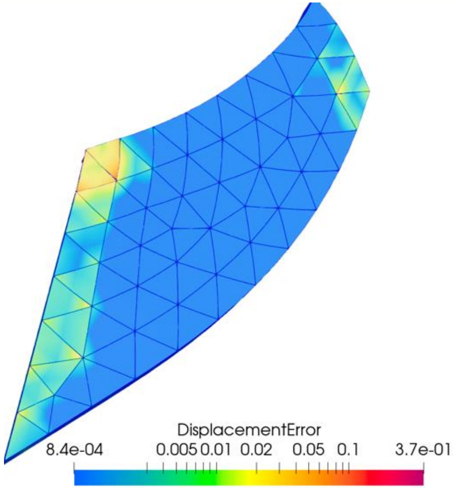
Displacement order 1



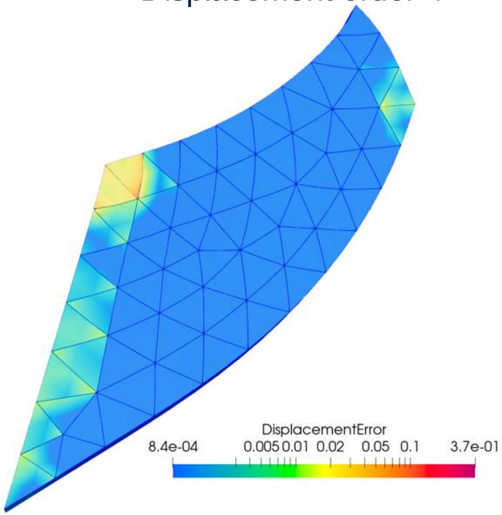
Displacement order 2



Displacement order 3



Displacement order 4



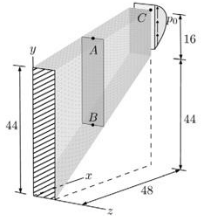
$$W_q(\mathbf{h}, \mathbf{h}, j) = \alpha_q(\mathbf{h} : \mathbf{h})^2 + \beta_q(\mathbf{y} : \mathbf{y})^2 + f_q(j)$$

$$f_q(j) = -24\beta \ln(j) - 12 ↵ \ln(j) + \frac{\lambda}{2''^2}(j^{''} + j^{-''})$$

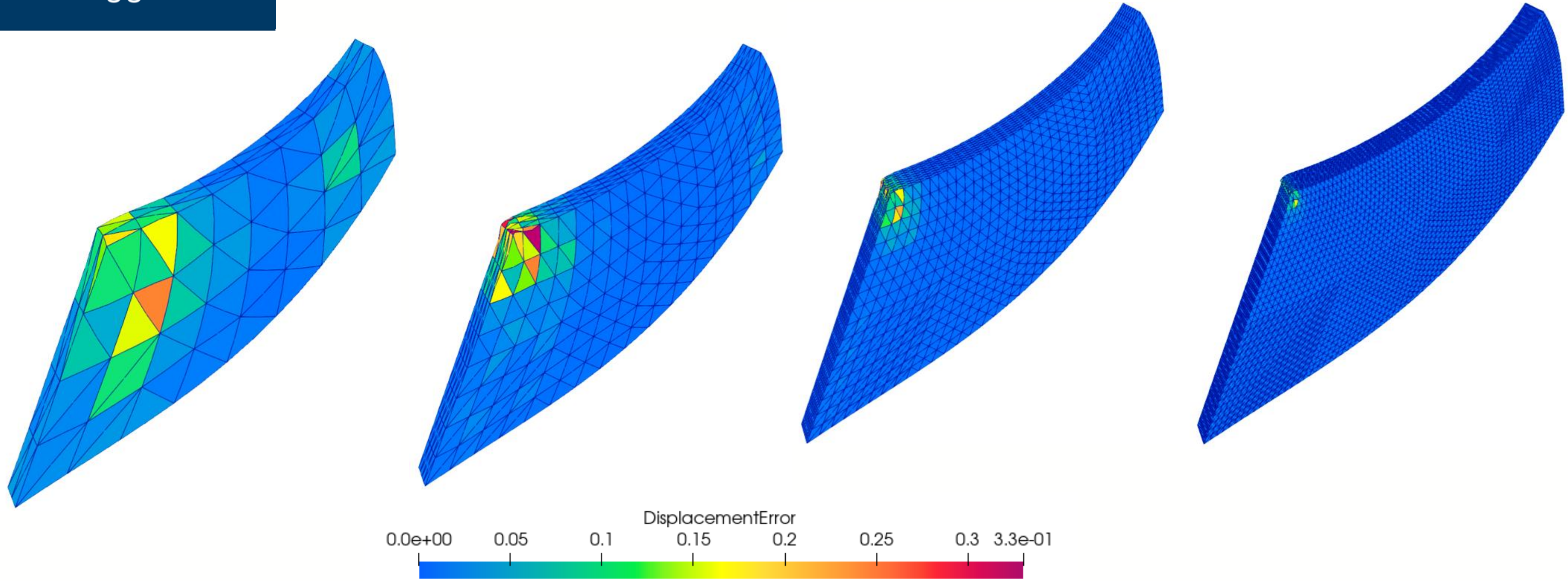
$$\alpha_q = 21\ \text{kPa}, \quad \beta_q = 42\ \text{kPa}, \quad \lambda_q = 8000\ \text{kPa}, \quad \varepsilon_q = 20.$$

Based on a test by Bonet et al., A computational framework for polyconvex large strain elasticity. Comput. Methods Appl. Mech. Eng. 283 (2015)

Schröder, J., Wriggers, P., & Balzani, D. (2011). A new mixed finite element based on different approximations of the minors of deformation tensors. *Computer Methods in Applied Mechanics and Engineering*, *200*(49-52), 3583-3600.



Cook beam (h-refinement)

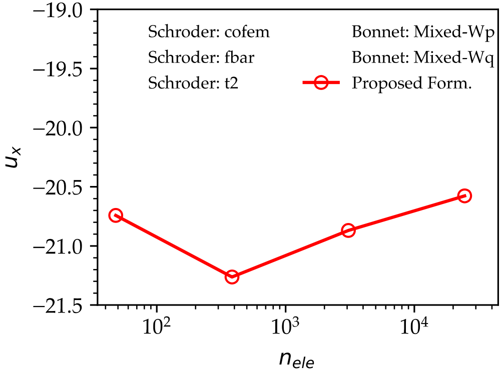


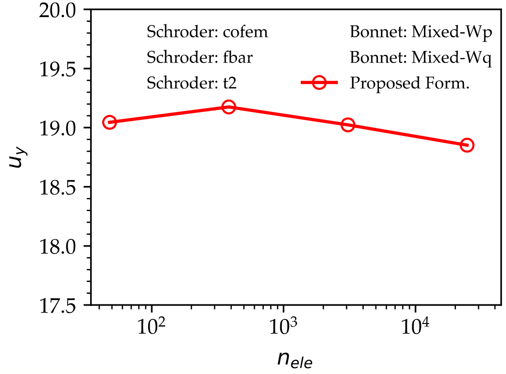
$$W_q(\mathbf{h}, \mathbf{h}, j) = \alpha_q(\mathbf{h} : \mathbf{h})^2 + \beta_q(\mathbf{y} : \mathbf{y})^2 + f_q(j)$$

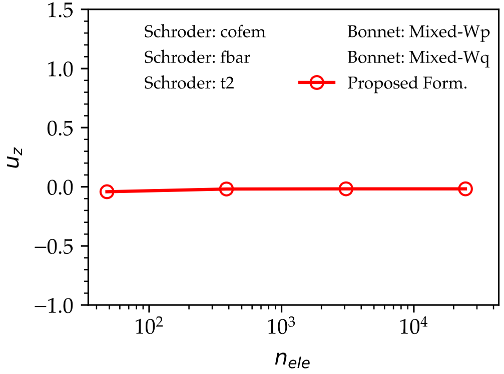
$$f_q(j) = -24\beta \ln(j) - 12\hookleftarrow \ln(j) + \frac{\lambda}{2''^2}(j^{''} + j^{-''})$$

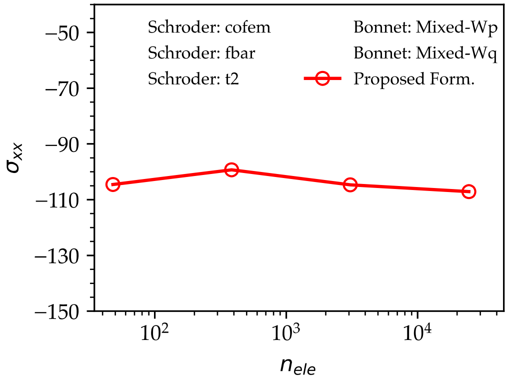
$\alpha_q = 21$ kPa, $\beta_q = 42$ kPa, $\lambda_q = 8000$ kPa, $\varepsilon_q = 20.$

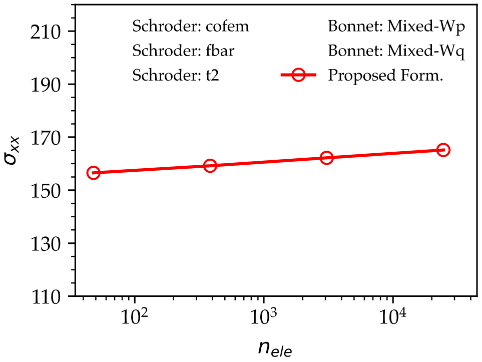
Cook beam: uniform mesh refinement study











Jörg Schröder, Peter Wriggers, and Daniel Balzani, “A New Mixed Finite Element Based on Different Approximations of the Minors of Deformation Tensors,” *Computer Methods in Applied Mechanics and Engineering*
Javier Bonet, Antonio J. Gil, and Rogelio Ortigosa, “A Computational Framework for Polyconvex Large Strain Elasticity,” *Computer Methods in Applied Mechanics and Engineering*

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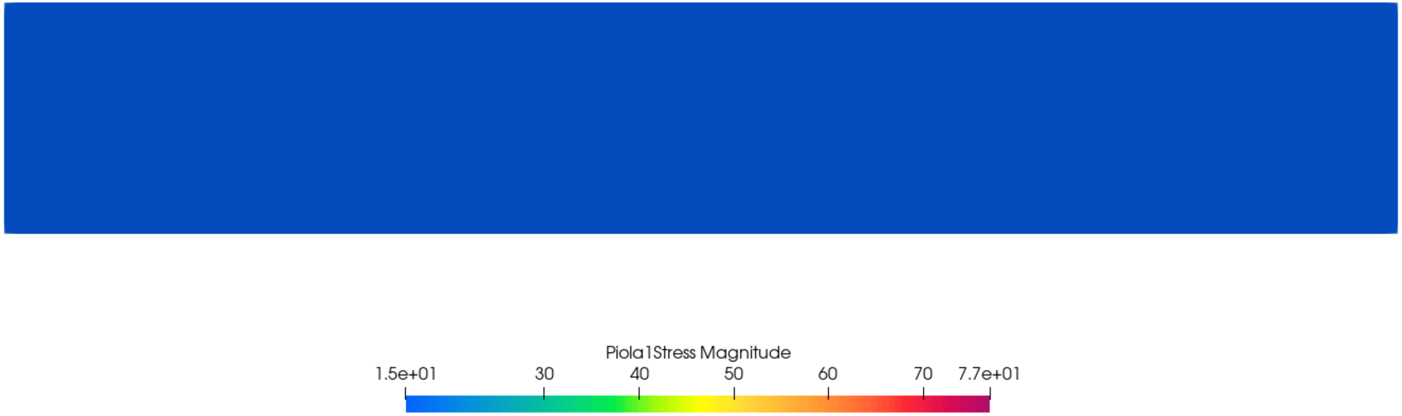
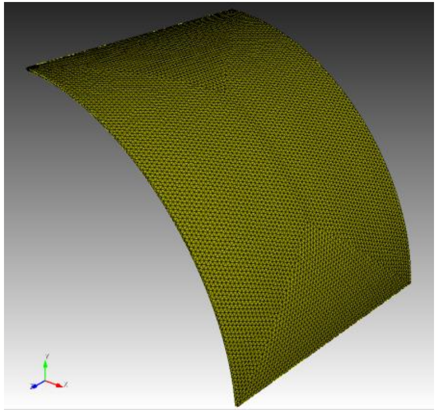
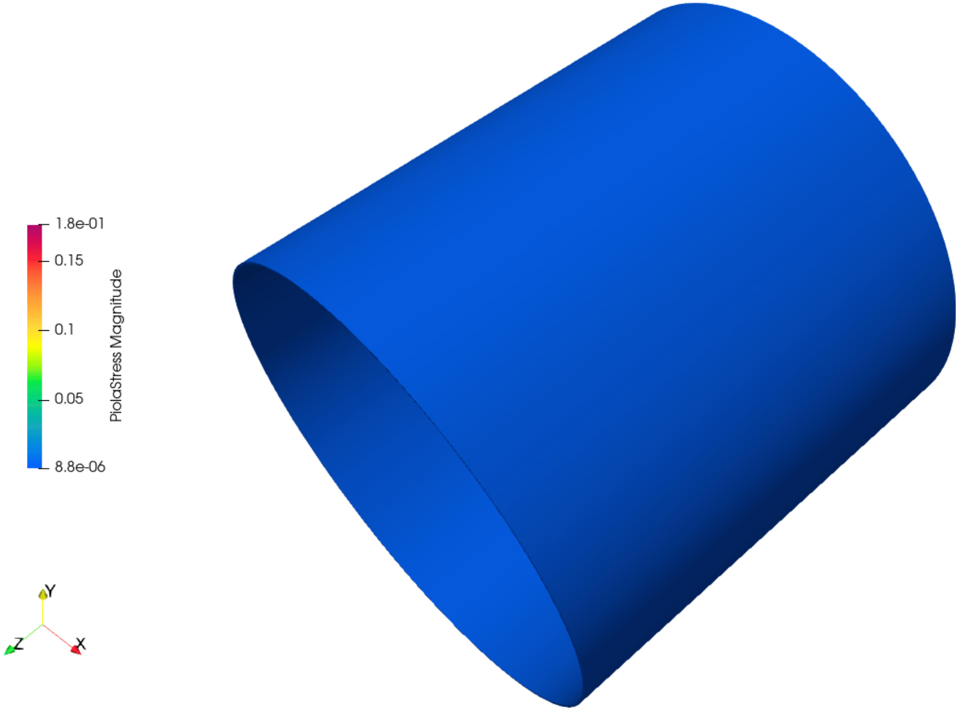
Twisting cantilever beam

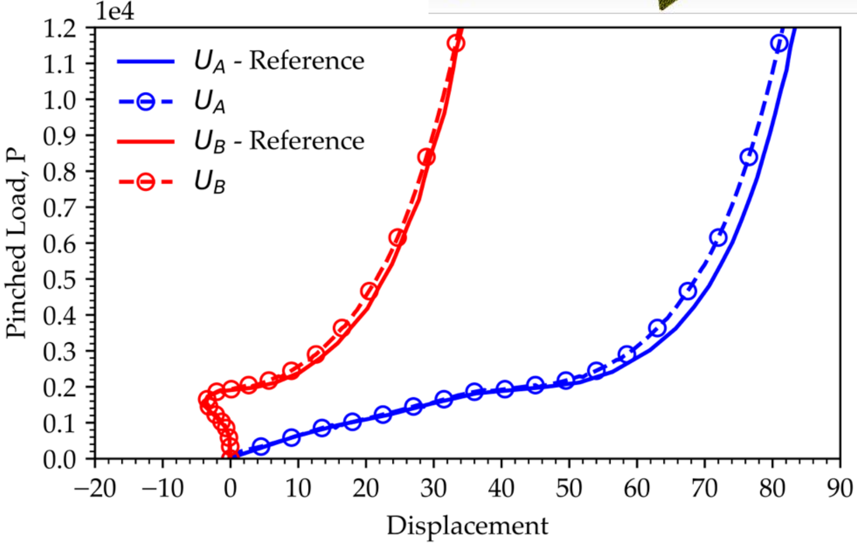
- Large rotations
- Neo-Hookean material–using new formulation.



Pinched cylindrical shell







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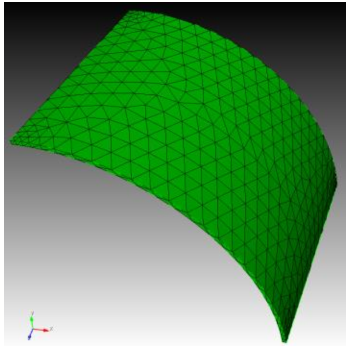
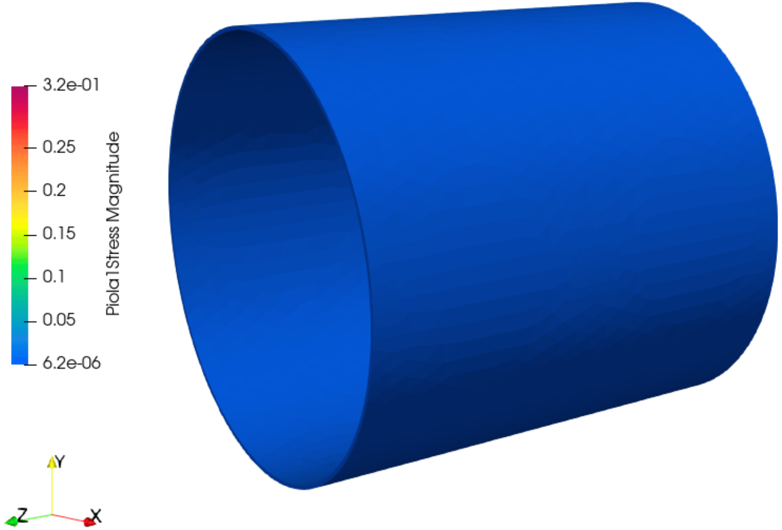
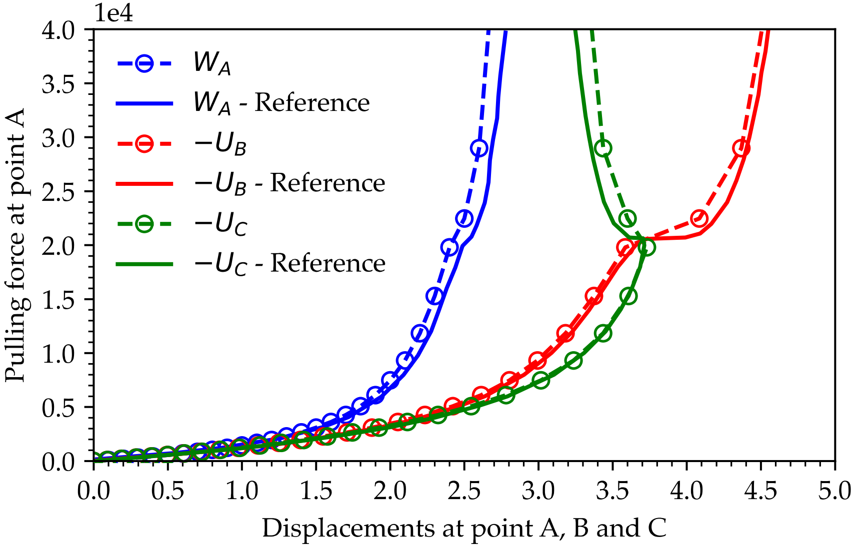
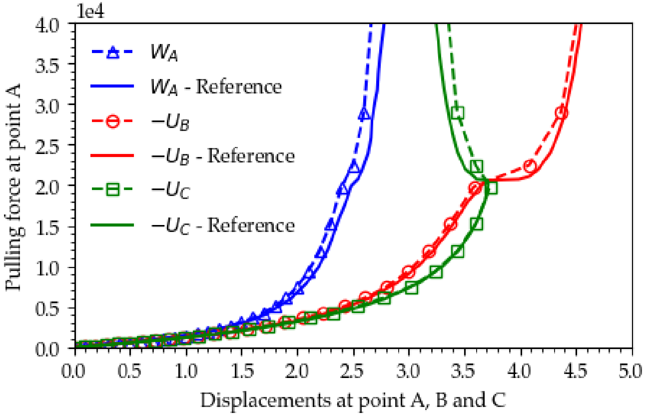
Pullout of an open-ended cylindrical shell



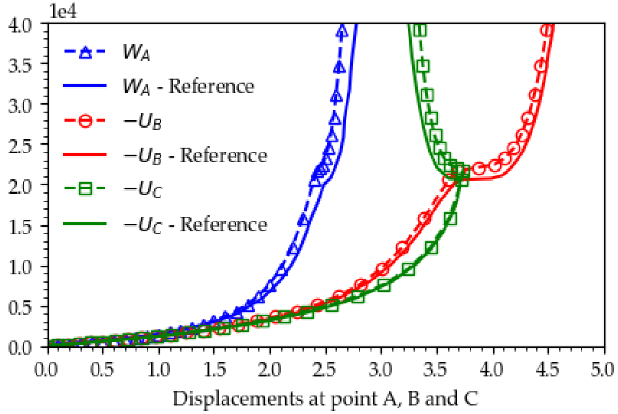




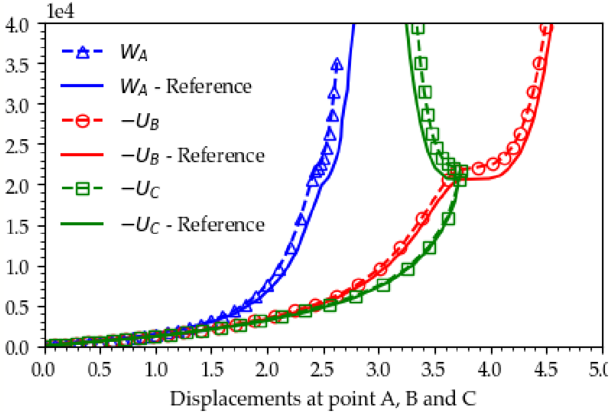
p-refinement



p=2

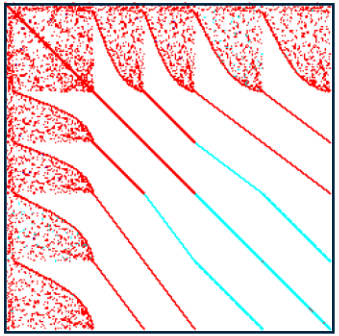


p=3



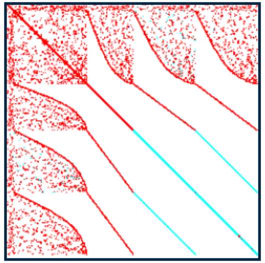
p=4

Exploiting block structure

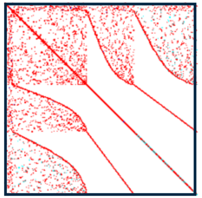


Each block corresponds to one FE

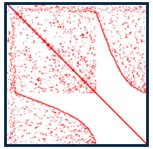
Eliminate logarithmic stretches

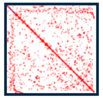


Eliminate normal stress bubbles

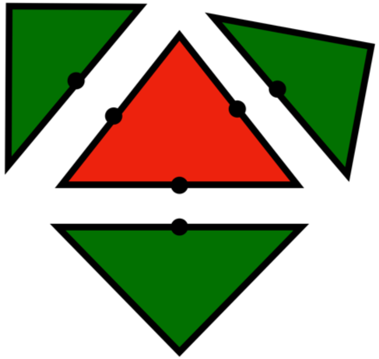


Eliminate rotation





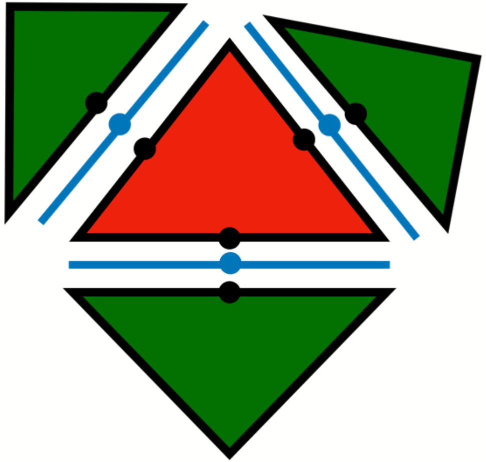
Three neighbors in 2d and 4 in 3D
(like quad mesh in finite volume method)



H-div data structure
Not well fit for of the shelf iterative solvers and their preconditioners
Multigrid is not trivial – Null Space – Neuman type problem

Hybridisation

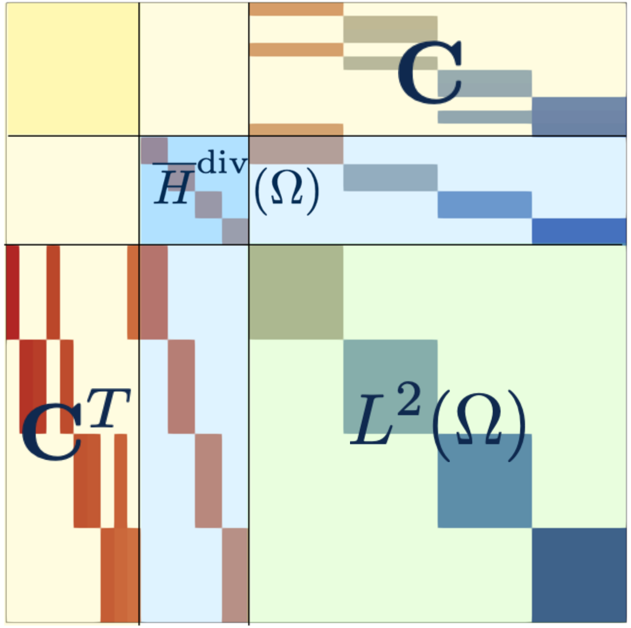
Three neighbors in 2d and 4 in 3D
(like quad mesh in finite volume method)



$$\overline{H}^{\mathrm{div}}(\Omega_0^h) := \left\{ \mathbf{P} \in \sum_e H^{\mathrm{div}}(\Omega_0^e) \right\}$$

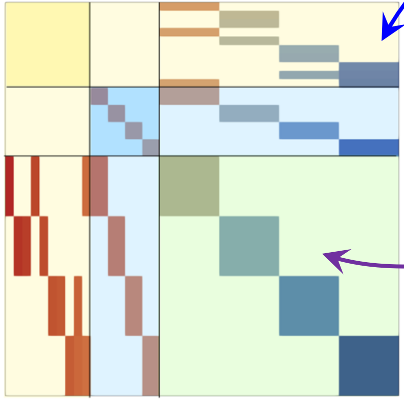
Boffi, Daniele, Franco Brezzi, and Michel Fortin. *Mixed finite element methods and applications*. Vol. 44. Heidelberg: Springer, 2013.

$$\int_{\Gamma^S} \delta u_i^S \left(\mathbf{N}_J^+ \mathbf{P}_{iJ}^+ + \mathbf{N}_J^- \cdot \mathbf{P}_{iJ}^- \right) \mathrm{d}\Gamma = 0$$



Dobrev, Veselin, et al. "Algebraic hybridization and static condensation with application to scalable H (div) preconditioning." *SIAM Journal on Scientific Computing* 41.3 (2019): B425-B447.

Block preconditioner – Exact Schur complement



$$\boxed{\mathbf{S} = -\mathbf{C}^T\mathbf{D}^{-1}\mathbf{C}}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f_u} \\ \mathbf{f_\lambda} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{S}^{-1} & -\mathbf{S}^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{S}^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}\mathbf{S}^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{f_u} \\ \mathbf{f_\lambda} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \boxed{\mathbf{D}^{-1}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{C} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boxed{\mathbf{S}^{-1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{B} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \boxed{\mathbf{D}^{-1}} \end{bmatrix} \begin{bmatrix} \mathbf{f_u} \\ \mathbf{f_\lambda} \end{bmatrix}$$

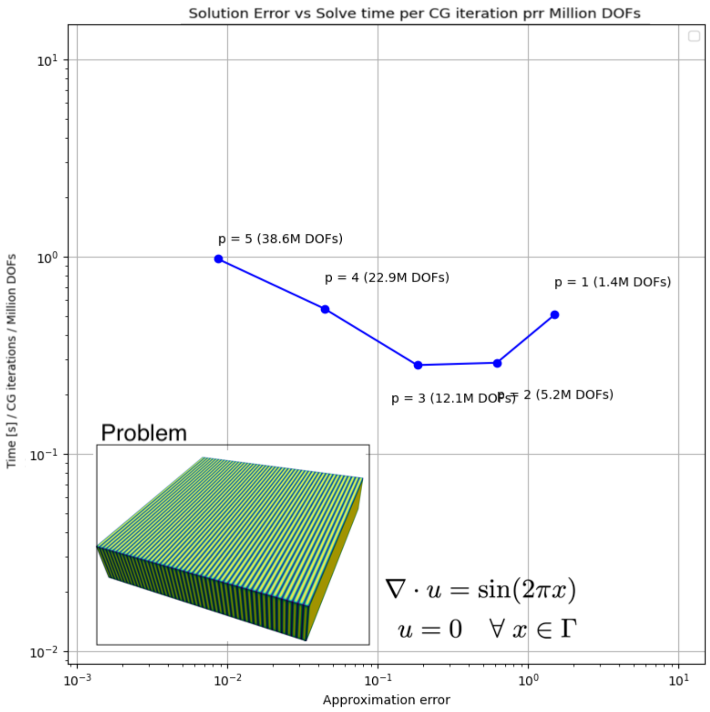
Use Algebraic Multigrid (AMG) to approximate S^{-1}

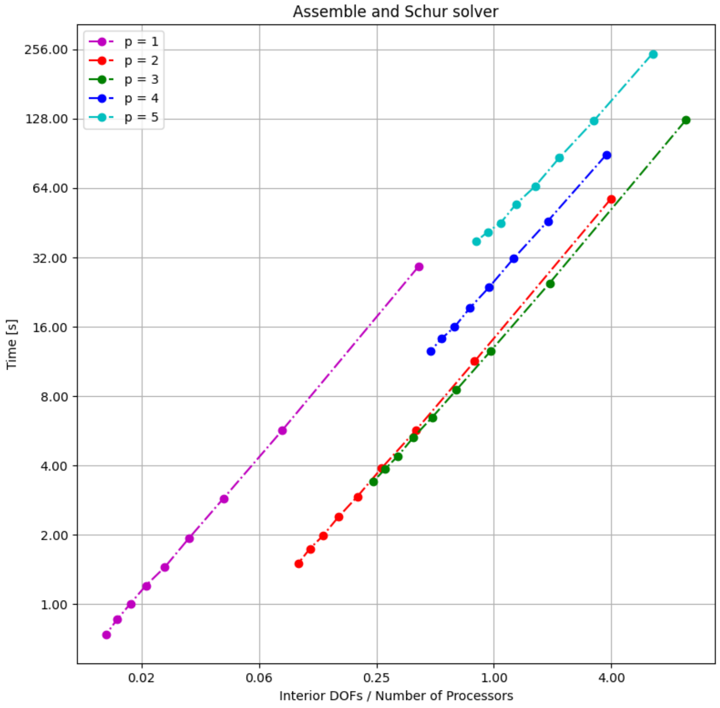
Invert exactly element-by-element

- ➢ Efficient block structure of Dense Block Matrices
- ➢ Block diagonal element by element
- ➢ Inverted exactly element by element
- ➢ Schur inversion is trivial
- ➢ Efficient to parallelize (GPU in perspective)

[10] PETSc/TAO Users Manual, https://petsc.org/

Hybridised solver





Stability of contact formulation

[3] Flemisch, B. and Wohlmuth, B.I. *Comput. Meth. Appl. Mech. Eng.*, *196*(8), 2007.
[4] Popp, A., et al., *SIAM J. Sci. Comput.*, 79(11), 2009.
[5] Boffi D, Brezzi F, Fortin M. Mixed finite element methods and applications (2013)

Stability of contact formulation

- Continuous functions $\mathcal{M} := H^1(\Gamma^c)$

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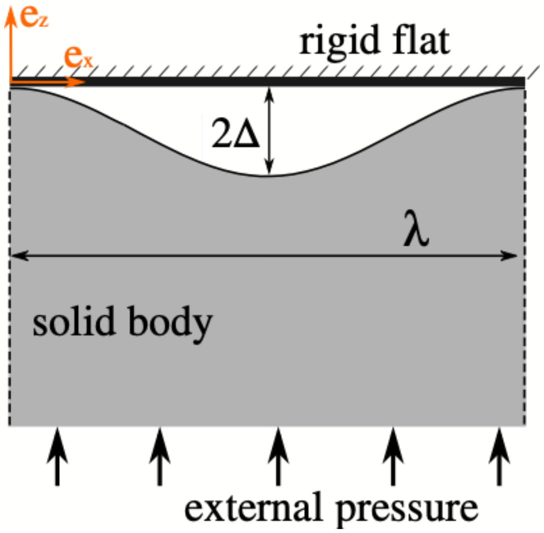
Test: 2D wavy surface contact



[3] Flemisch, B. and Wohlmuth, B.I. *Comput. Meth. Appl. Mech. Eng.*, *196*(8), 2007.
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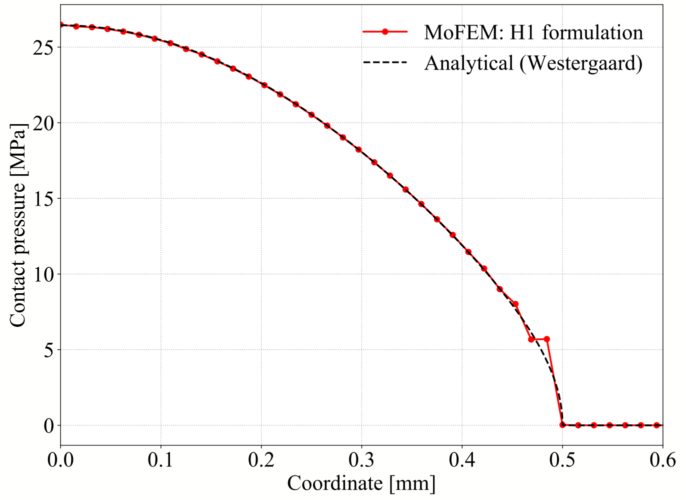

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Stability of contact formulation

- Continuous functions $\mathcal{M} := H^1(\Gamma^c)$
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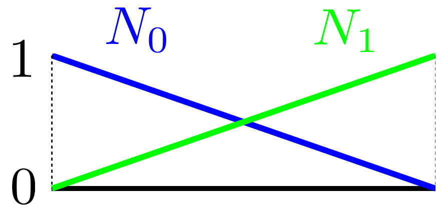
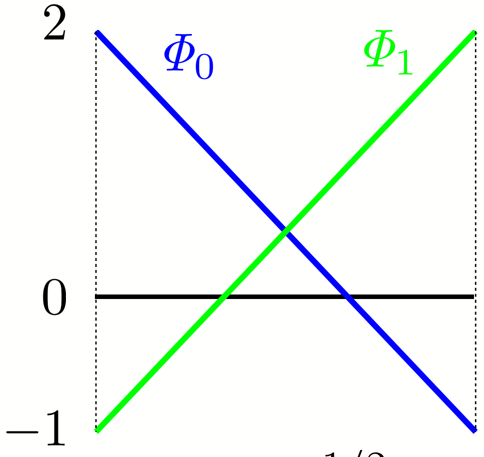
  $\text{trace} : H^1(\Omega) \to H^{1/2}(\Gamma^c)$

  dual space for $H^{1/2}(\Gamma^c)$ is denoted as $H^{-1/2}(\Gamma^c)$



$N_M \in H^{1/2}(\Gamma_c)$



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[3] Flemisch, B. and Wohlmuth, B.I. *Comput. Meth. Appl. Mech. Eng.*, *196*(8), 2007.
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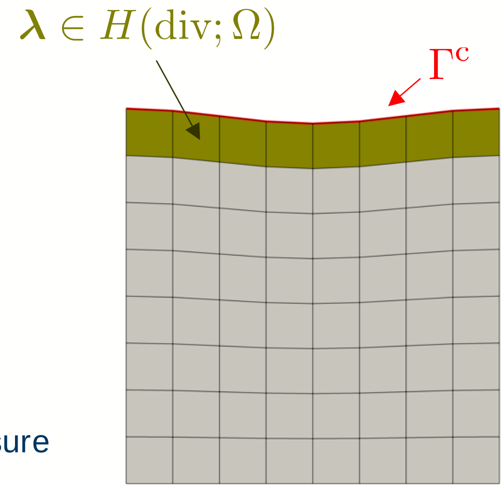
- Raviart-Thomas space [5]

$$H(\text{div};\Omega) := \left\{ \mathbf{u} \in \left[L^2(\Omega)\right]^n \mid \text{div}\, \mathbf{u} \in L^2(\Omega) \right\}$$

$$\text{normal trace} : \boldsymbol{\lambda} \in \underbrace{H(\text{div};\Omega)}_{\text{natural space for stress}} \rightarrow \boldsymbol{\lambda} \cdot \mathbf{n}|_{\Gamma^c} \in \underbrace{H^{-1/2}(\Gamma^c)}_{\text{natural space for pressure}}$$



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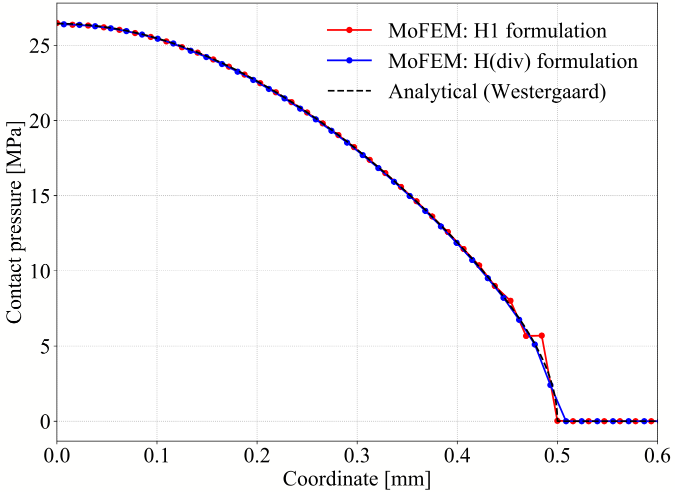

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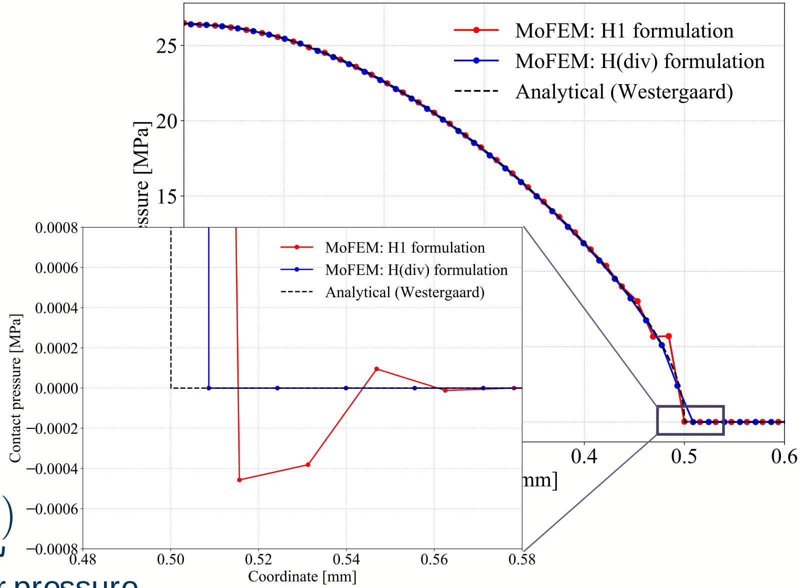

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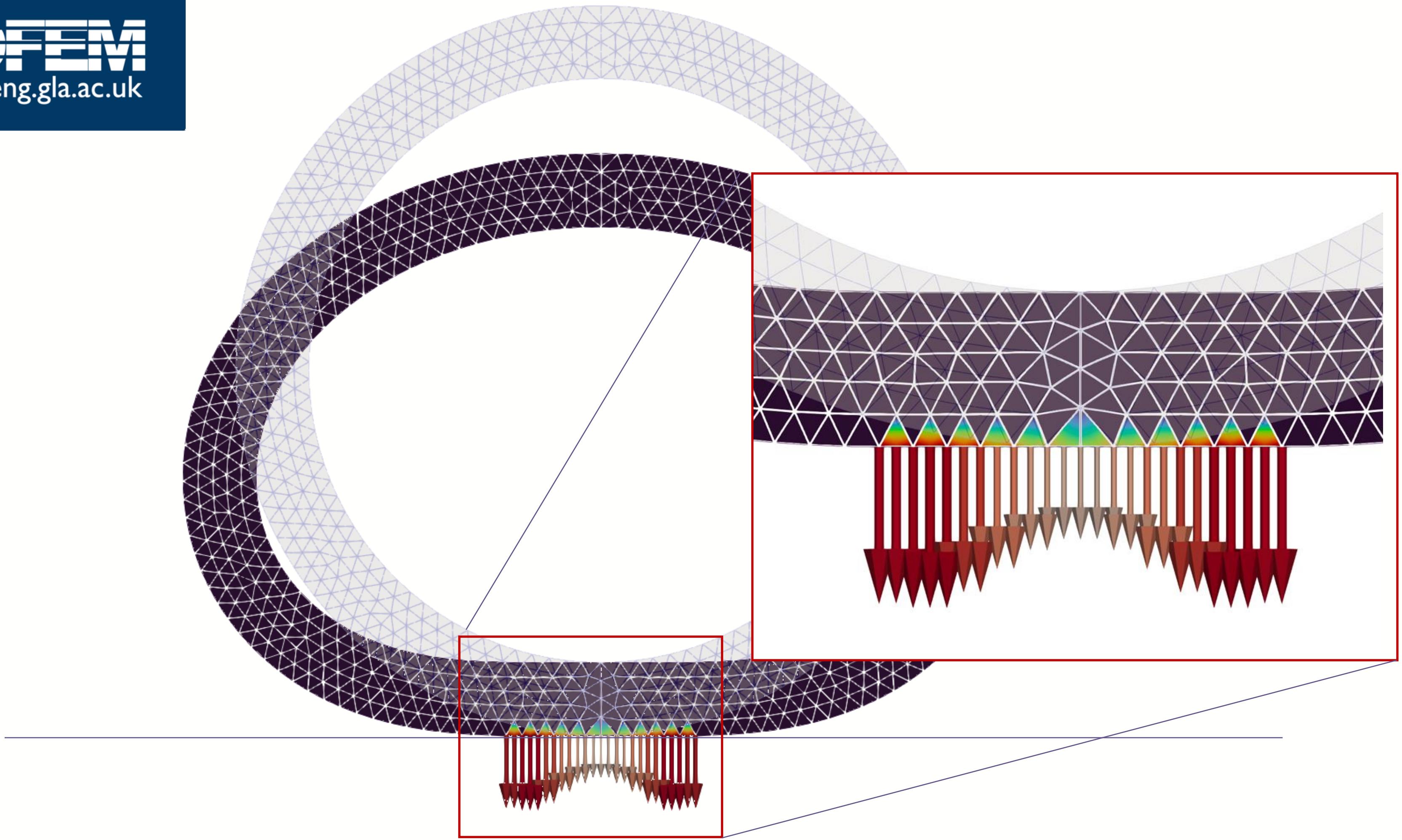
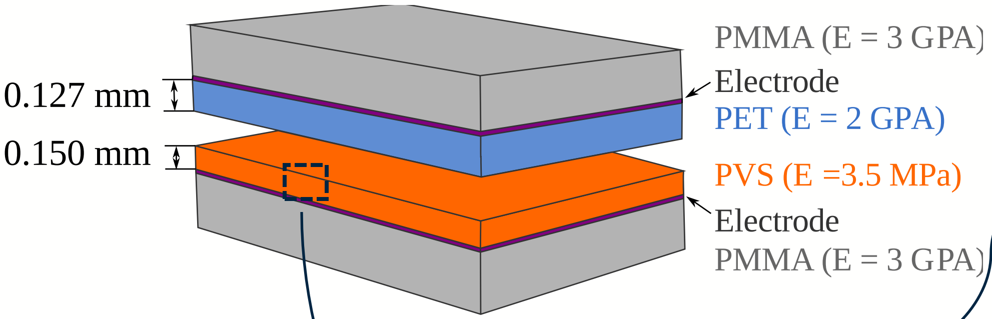
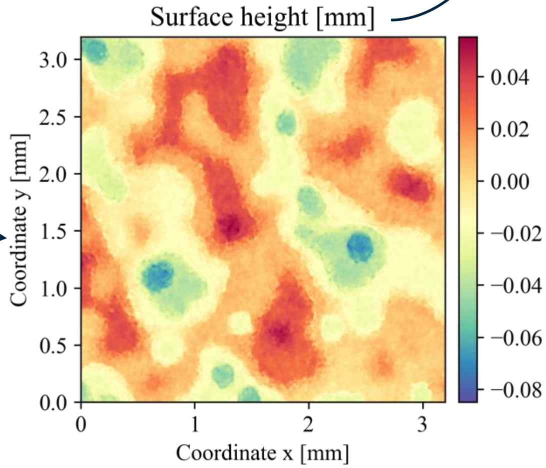
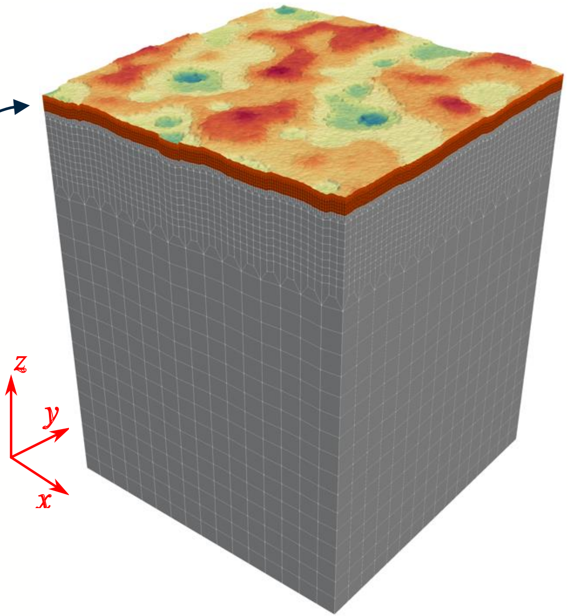
MOFEM
mofem.eng.gla.ac.uk



Stability is essential for contact area estimation







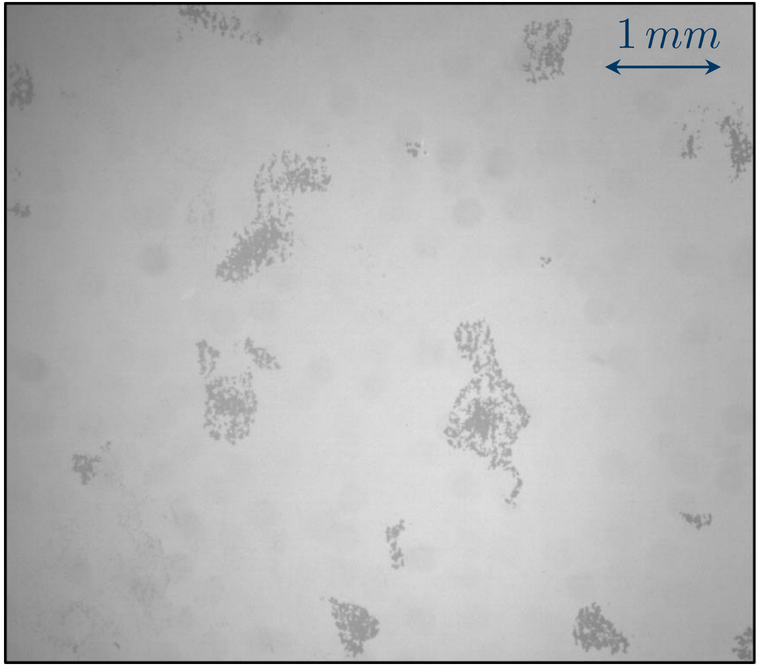
Andrei Shvarts

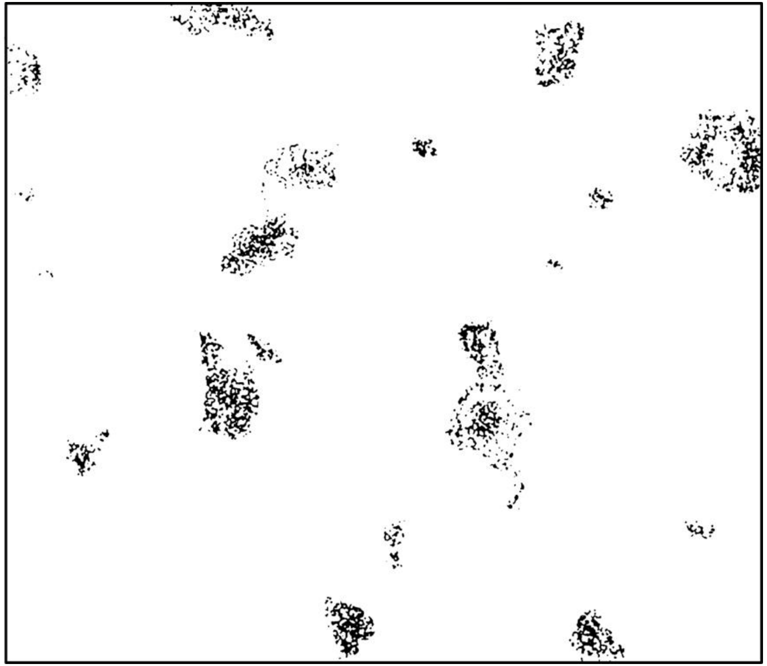
MD Tanzib Ehsan Sanglap

$S_q \approx 0.025$ mm

$S_{dq} \approx 0.3$

Contact area morphology (real roughness)



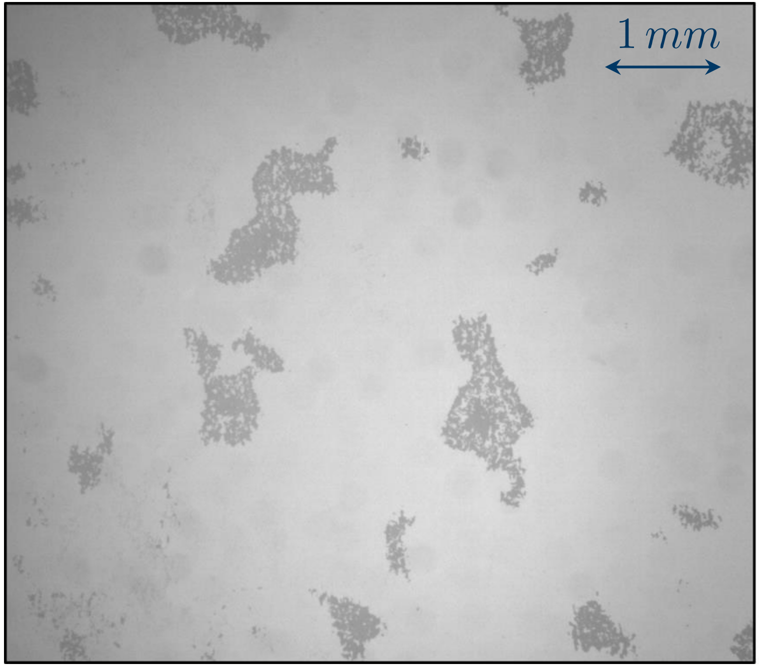


Experiment (interference reflection microscopy)

[8] Kumar C. et al. *Nano Energy* 107 (2023)

$p_{\mathrm{ext}} \approx 0.01\,\mathrm{MPa}$ Simulation

Contact area morphology (real roughness)



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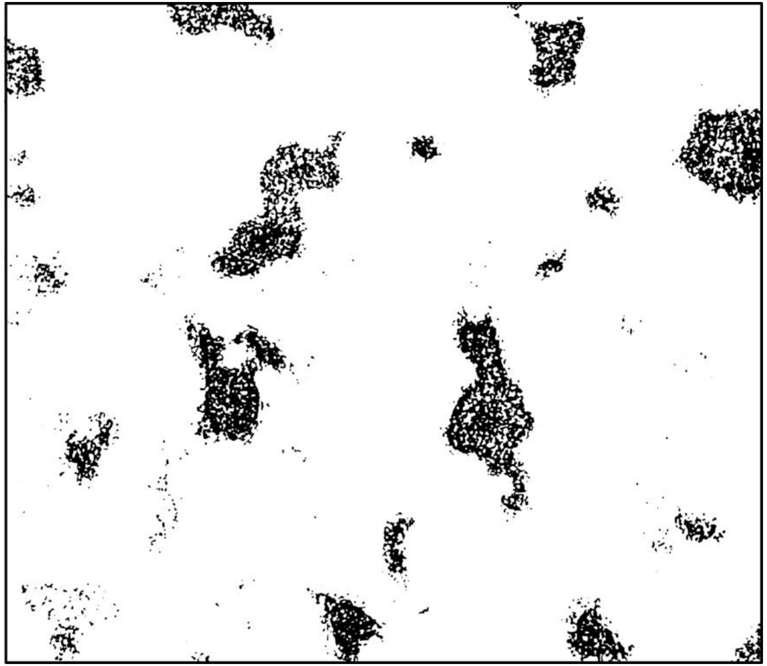
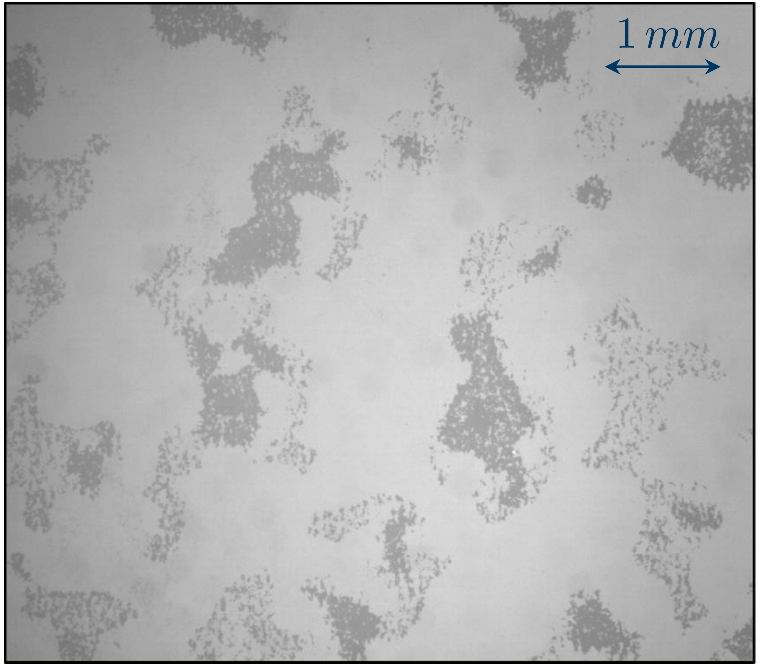


$p_{\text{ext}} \approx 0.04\,$MPa Simulation

Contact area morphology (real roughness)



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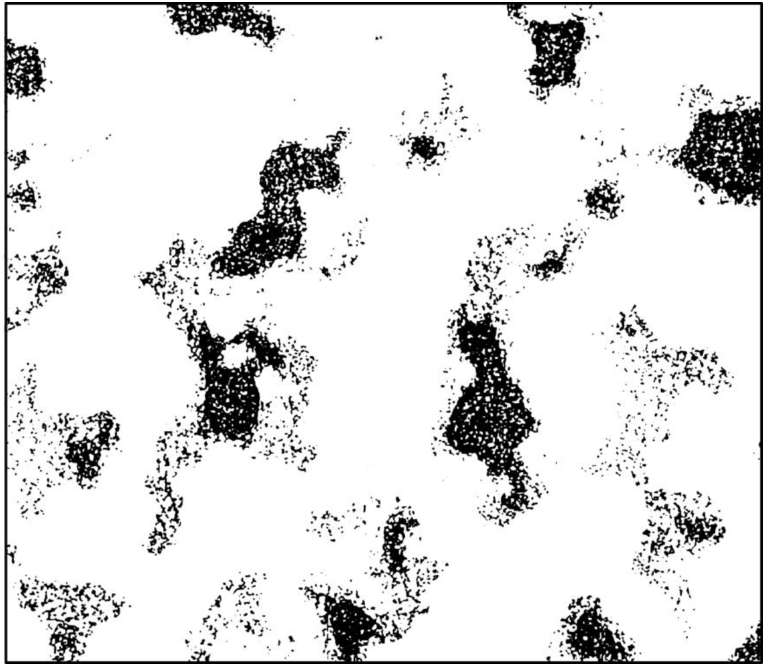
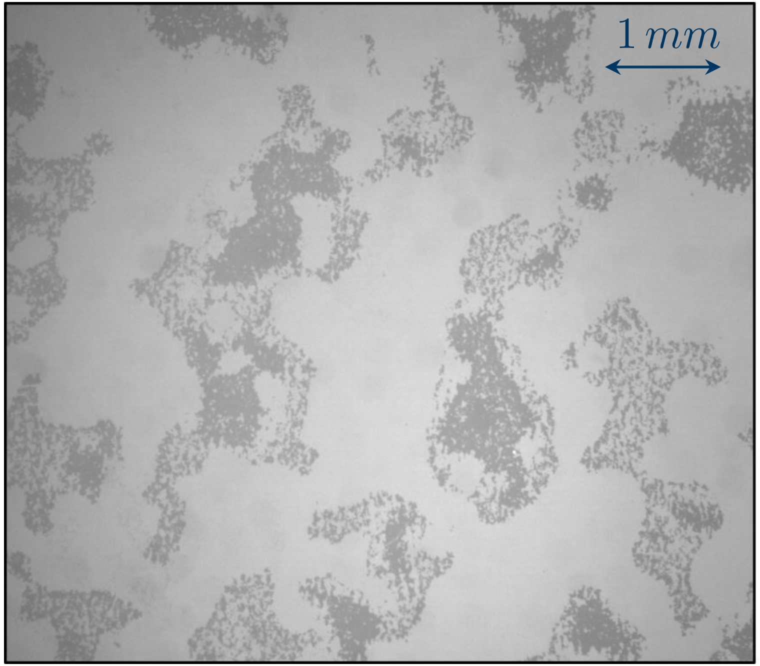


$p_{\text{ext}} \approx 0.07\,\text{MPa}$ Simulation

Contact area morphology (real roughness)



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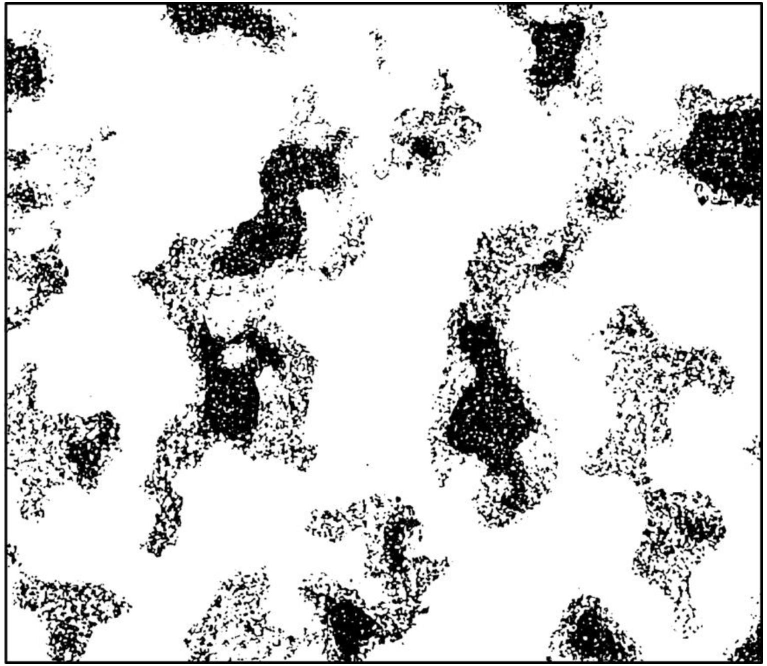
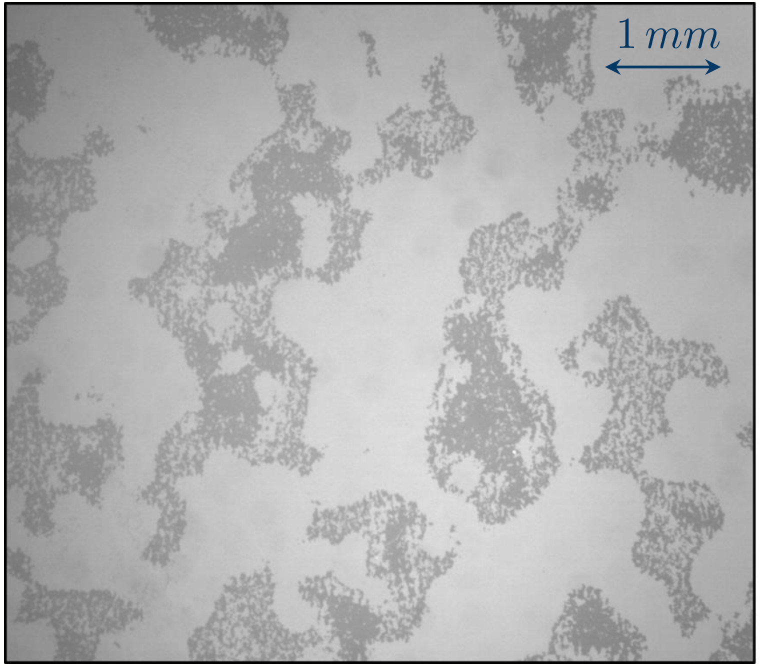
[8] Kumar C. et al. *Nano Energy* 107 (2023)



$p_{\text{ext}} \approx 0.1\,$MPa

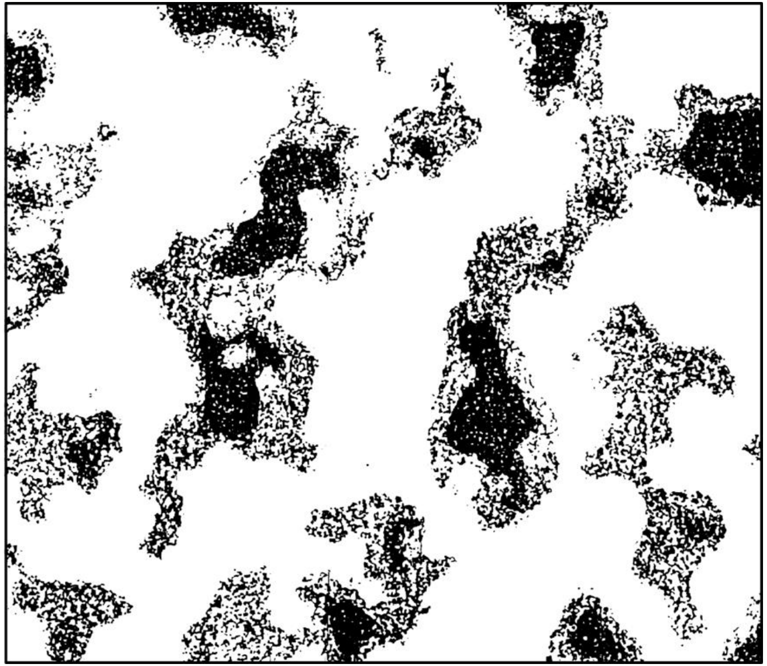
Simulation

Contact area morphology (real roughness)



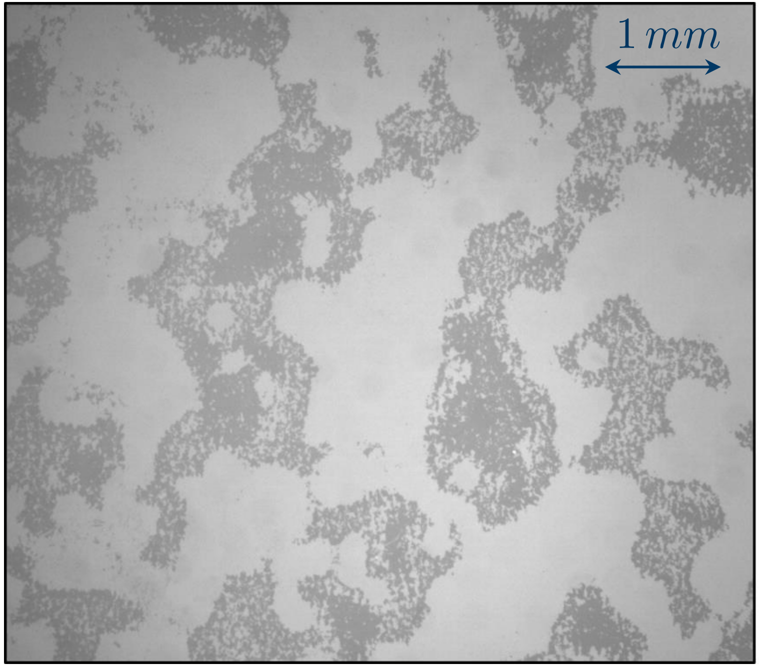
Experiment (interference reflection microscopy)

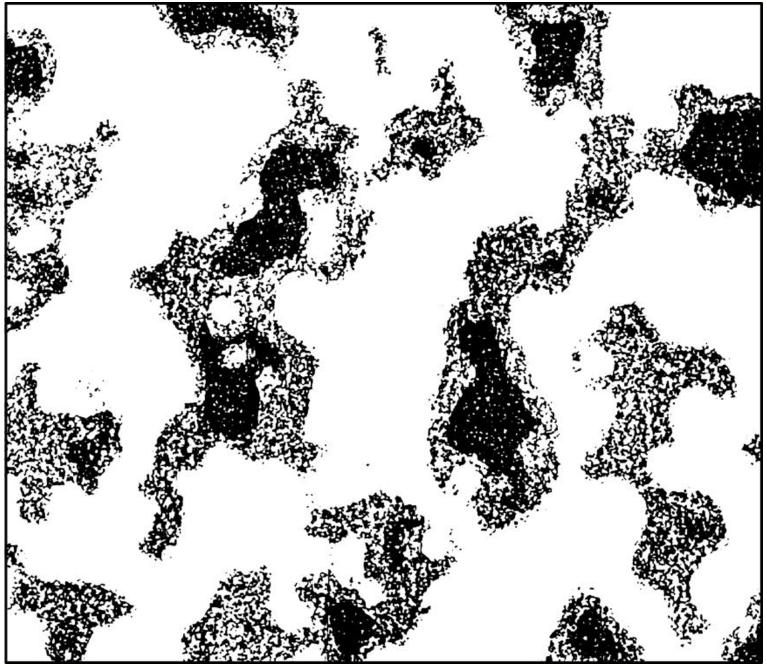
[8] Kumar C. et al. *Nano Energy* 107 (2023)



$p_{\text{ext}} \approx 0.13\,\text{MPa}$ Simulation

Contact area morphology (real roughness)





Experiment (interference reflection microscopy)

[8] Kumar C. et al. *Nano Energy* 107 (2023)

$p_{\text{ext}} \approx 0.16\,\text{MPa}$ Simulation

Signed distance function

Signed distance function defined for a domain Ω outside the surface

$$s(\boldsymbol{x}) = \begin{cases} d(\boldsymbol{x}, \partial\Omega) & \text{if } \boldsymbol{x} \in \Omega \\ -d(\boldsymbol{x}, \partial\Omega) & \text{if } \boldsymbol{x} \notin \Omega \end{cases}$$

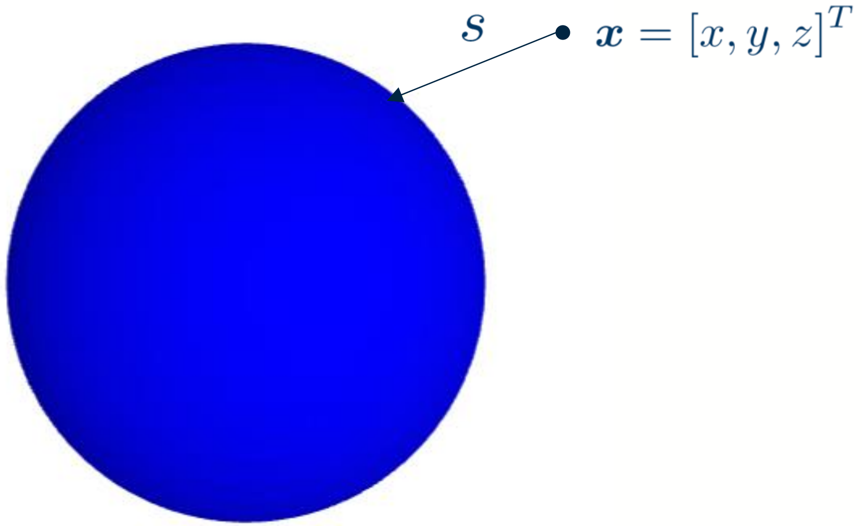
$$||\nabla d(\boldsymbol{x}, \partial\Omega)|| = 1$$

Sphere with radius r and centroid:

$$\boldsymbol{x_c} = [x_c, y_c, z_c]^T$$

Signed distance function for a sphere:

$$s = \sqrt{(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2} - r$$



Contact formulation

One if positive gap, or positive traction, i.e. no contact

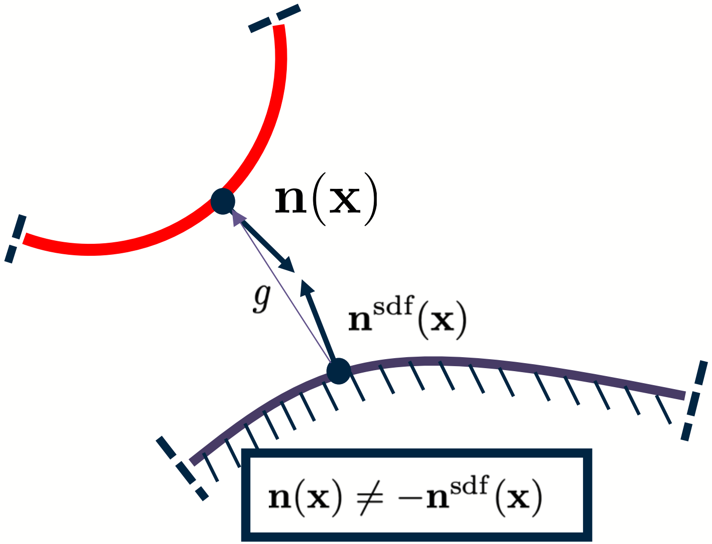
$$\mathcal{C}(g, t_n) = \frac{1}{2}(1 - \text{sign}(g - c_n t_n))$$

$$P^{\mathcal{C}}_{ij} = \mathcal{C}(n^{\text{sdf}}_i n^{\text{sdf}}_j)$$ ← Contact projection operator

$$Q^{\mathcal{C}}_{ij} = \delta_{ij} - P^{\mathcal{C}}_{ij}$$ ← Tangent projection operator

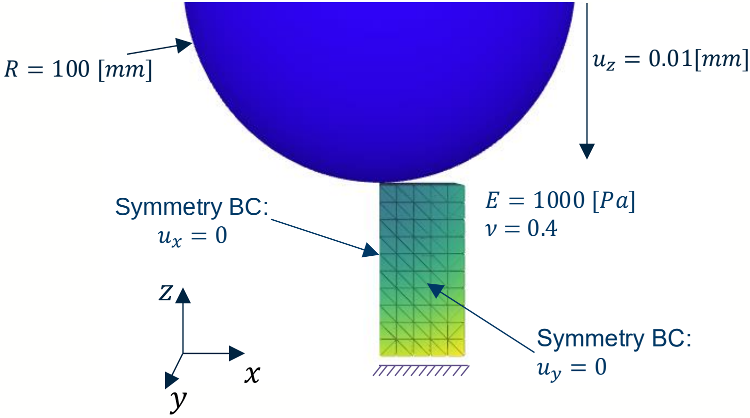
$$f^{\mathcal{S}} = \int_{\Gamma^c} \delta u^{\mathcal{S}}_i \left(Q^{\mathcal{C}}_{ij} \left(N_J P_{jJ}\right) + \frac{\mathcal{C}}{c_n} n^{\text{sdf}}_i g \right) \mathrm{d}\Gamma$$

$$f^{\mathcal{P}} = \int_{\Gamma^c} \left(N_J \delta P_{iJ}\right) u^{\mathcal{S}}_i \mathrm{d}\Gamma$$

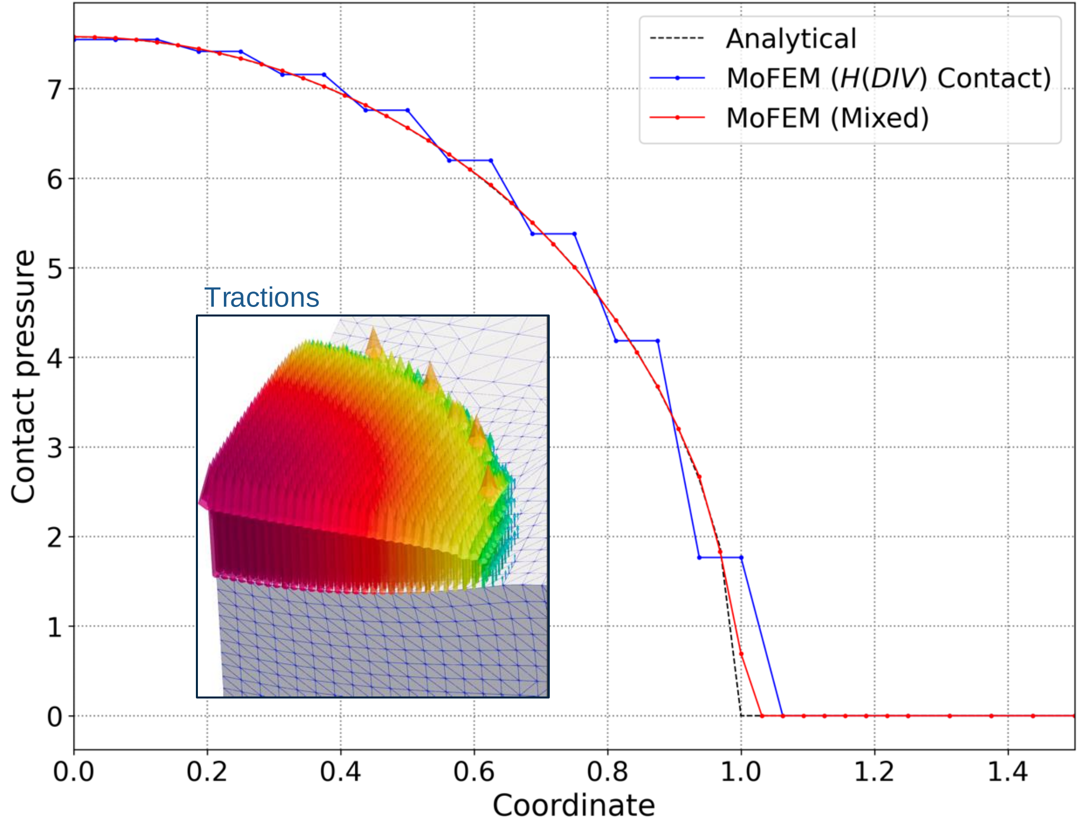


Note: Hybridised displacement on skeleton (contact boundary is part of it), is a Lagrange multiplier enforcing contact constraints.

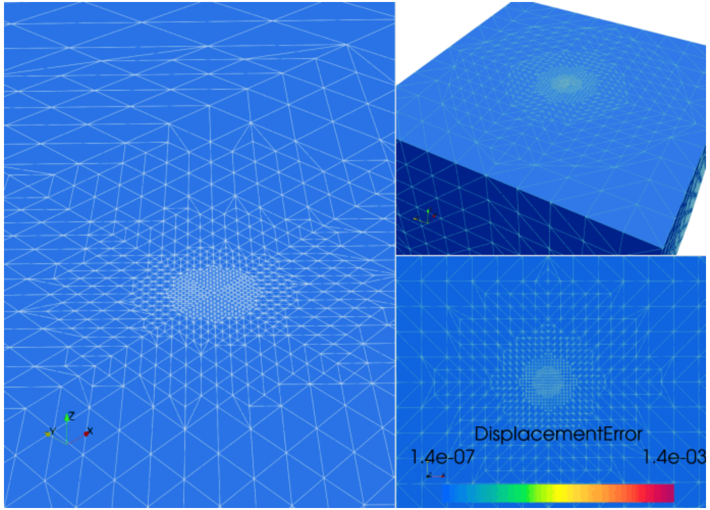
Hertz contact problem



Result for displacements order 2:



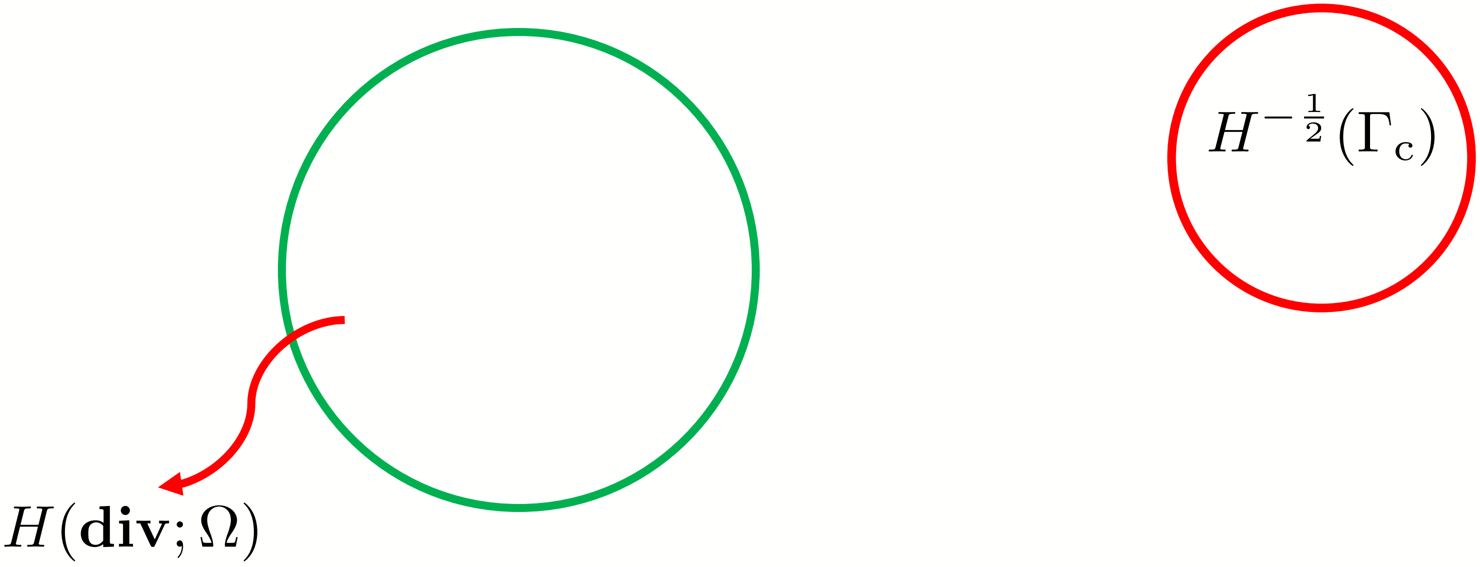
Displacement error



Restricting H(div; Ω) to RT

Boffi et al., 2013: **Lemma 2.1.2.**

The normal trace operator γ_n: $\boldsymbol{\lambda} \in H(\mathbf{div};\Omega) \rightarrow \boldsymbol{\lambda}\cdot\mathbf{n}|_\Gamma \in H^{-1/2}(\Gamma)$ is surjective



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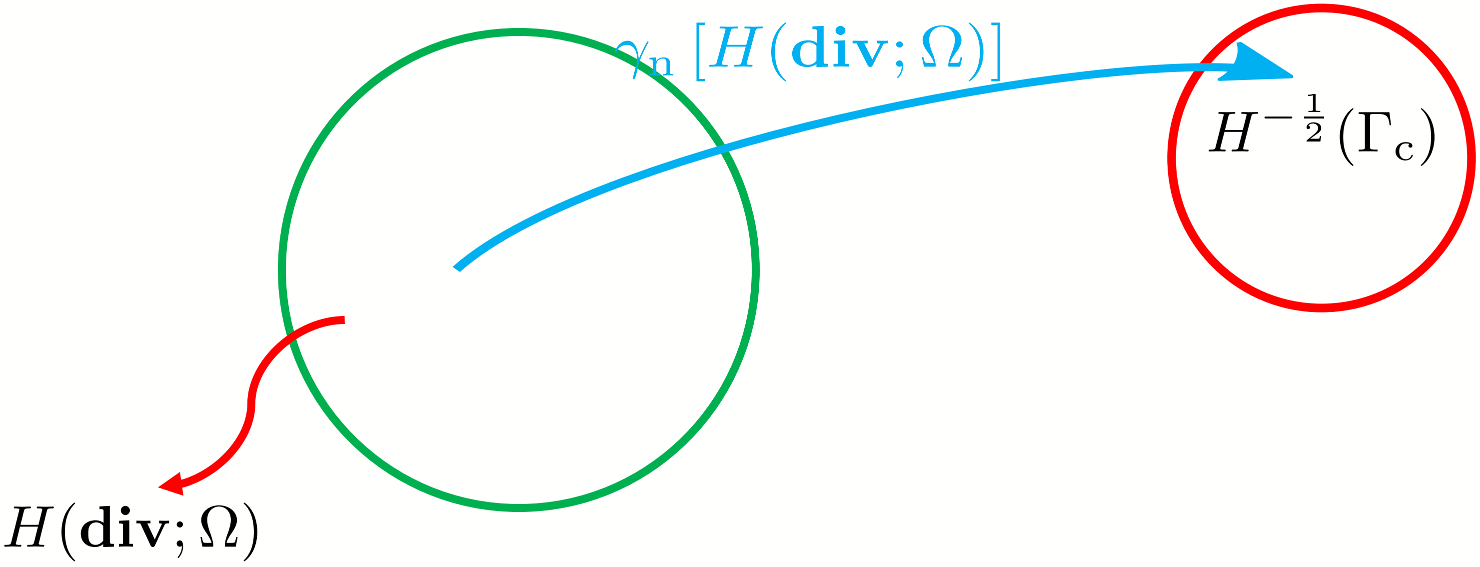

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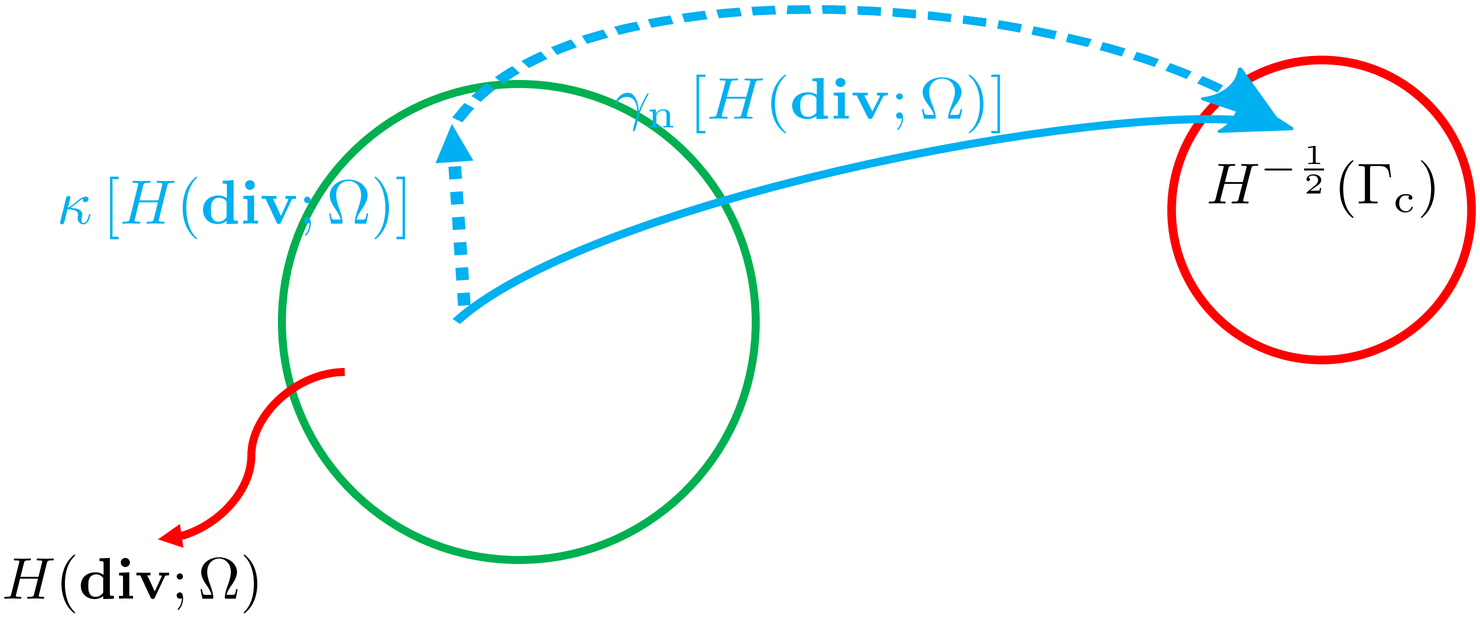

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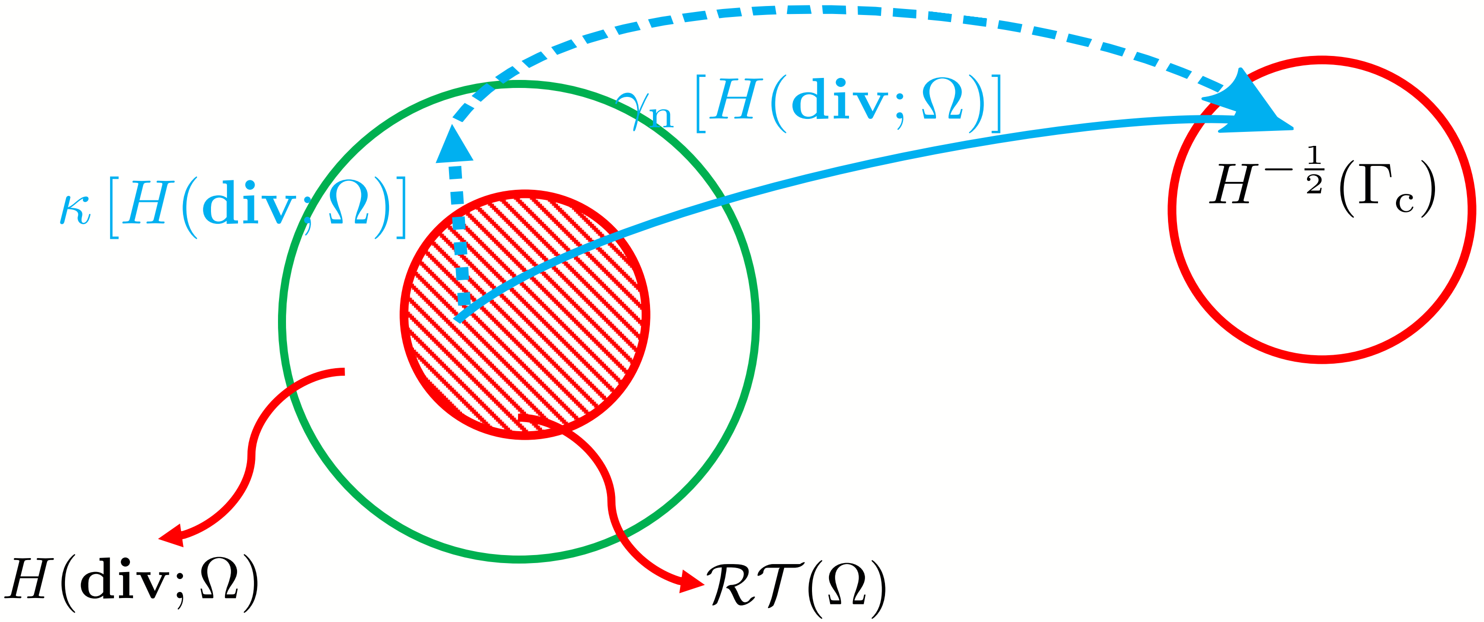

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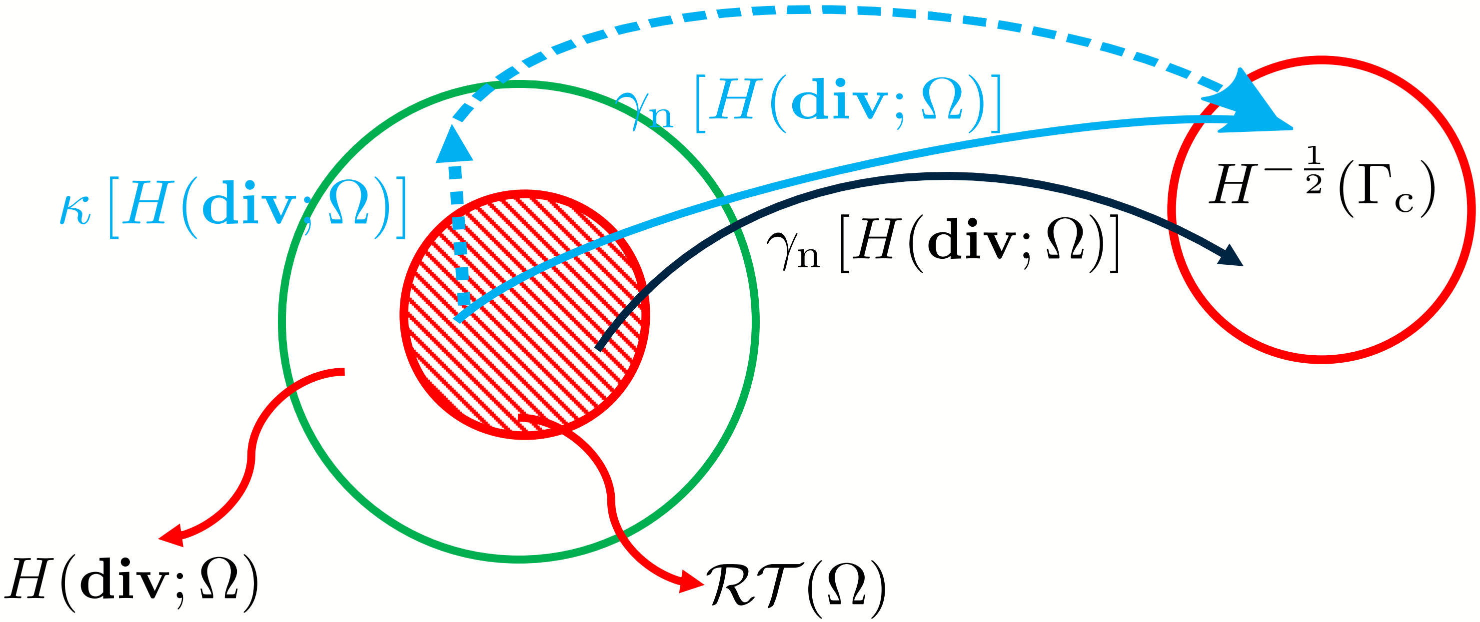

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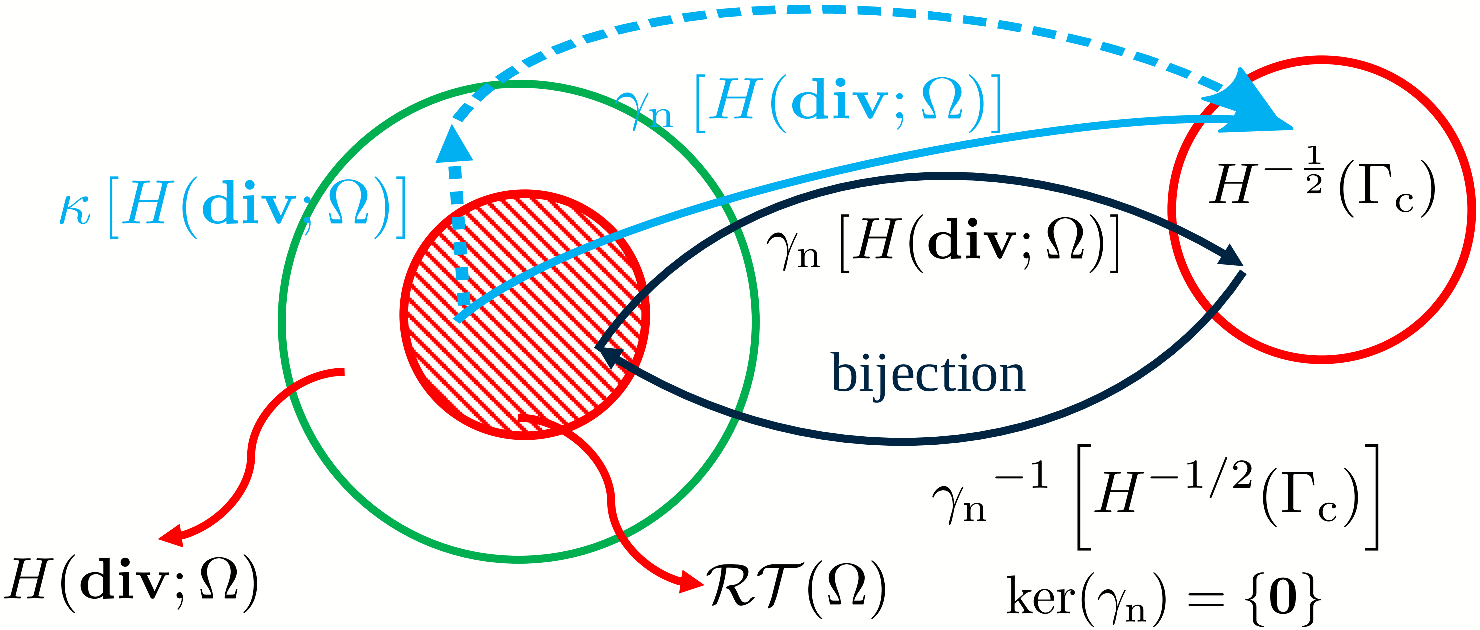
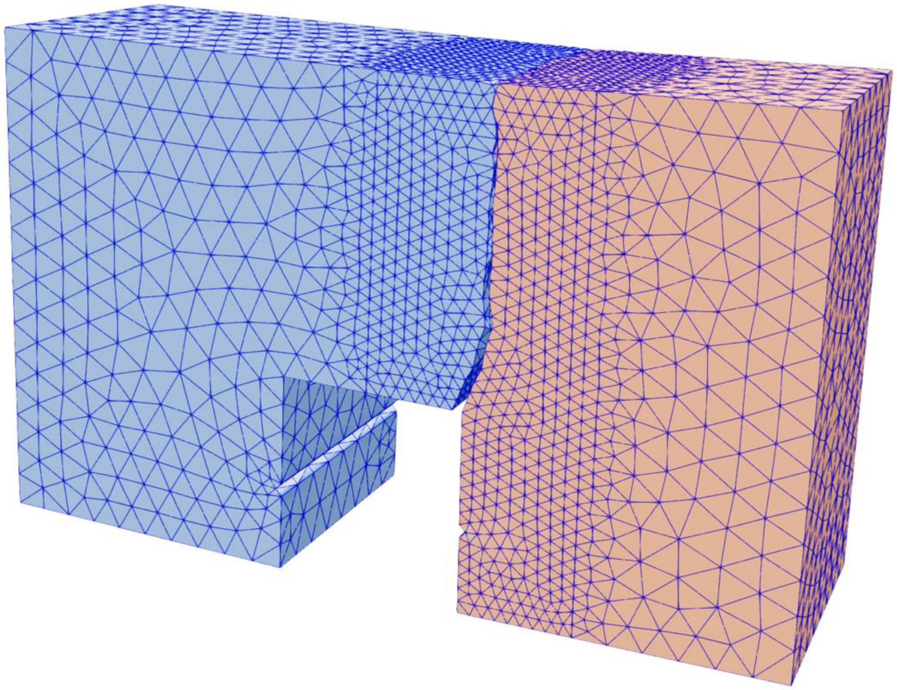


Fracture is natural, for weakly enforced conformity

Color represents z-axis rotation



- Crack propagates by erasing rows and columns of the matrix. Matrix adjacency is fixed. That provides robustness
- Trace of H-div space is associated with faces, thus crack face energy can be easily estimated
- If crack propagate one face by one face, and iterative solver is deployed, crack propagation are resolved on linear solver level

$$\Psi^{\mathcal{F}} = \int_{\Omega} \sigma^{\mathcal{F}}_{ij} \varepsilon_{ij} \mathrm{d}\Omega$$

Extension of face trace

University of Glasgow

Thank you for your attention!

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